CHAPTER 4

Solution to Problem 4.1. Let $Y = \sqrt{|X|}$. We have, for $0 \le y \le 1$,

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}(\sqrt{|X|} \le y) = \mathbf{P}(-y^2 \le X \le y^2) = y^2,$$

and therefore by differentiation,

$$f_Y(y) = 2y, \qquad \text{for } 0 \le y \le 1.$$

Let $Y = -\ln|X|$. We have, for $y \ge 0$,

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}(\ln|X| \ge -y) = \mathbf{P}(X \ge e^{-y}) + \mathbf{P}(X \le -e^{-y}) = 1 - e^{-y},$$

and therefore by differentiation

$$f_Y(y) = e^{-y}, \qquad \text{for } y \ge 0,$$

so Y is an exponential random variable with parameter 1. This exercise provides a method for simulating an exponential random variable using a sample of a uniform random variable.

Solution to Problem 4.2. Let $Y = e^X$. We first find the CDF of Y, and then take the derivative to find its PDF. We have

$$\mathbf{P}(Y \le y) = \mathbf{P}(e^X \le y) = \begin{cases} \mathbf{P}(X \le \ln y), & \text{if } y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$f_Y(y) = \begin{cases} \dfrac{d}{dx} F_X(\ln y), & \text{if } y > 0, \\ 0, & \text{otherwise,} \end{cases}$$
$$= \begin{cases} \dfrac{1}{y} f_X(\ln y), & \text{if } y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

When X is uniform on $[0, 1]$, the answer simplifies to

$$f_Y(y) = \begin{cases} \dfrac{1}{y}, & \text{if } 0 < y \le e, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 4.3. Let $Y = |X|^{1/3}$. We have

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}\big(|X|^{1/3} \le y\big) = \mathbf{P}\big(-y^3 \le X \le y^3\big) = F_X(y^3) - F_X(-y^3),$$

and therefore, by differentiating,

$$f_Y(y) = 3y^2 f_X(y^3) + 3y^2 f_X(-y^3), \qquad \text{for } y > 0.$$

Let $Y = |X|^{1/4}$. We have

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}\big(|X|^{1/4} \le y\big) = \mathbf{P}(-y^4 \le X \le y^4) = F_X(y^4) - F_X(-y^4),$$

and therefore, by differentiating,

$$f_Y(y) = 4y^3 f_X(y^4) + 4y^3 f_X(-y^4), \qquad \text{for } y > 0.$$

Solution to Problem 4.4. We have

$$F_Y(y) = \begin{cases} 0, & \text{if } y \le 0, \\ \mathbf{P}(5 - y \le X \le 5) + \mathbf{P}(20 - y \le X \le 20), & \text{if } 0 \le y \le 5, \\ \mathbf{P}(20 - y \le X \le 20), & \text{if } 5 < y \le 15, \\ 1, & \text{if } y > 15. \end{cases}$$

Using the CDF of X, we have

$$\mathbf{P}(5 - y \le X \le 5) = F_X(5) - F_X(5 - y),$$

$$\mathbf{P}(20 - y \le X \le 20) = F_X(20) - F_X(20 - y).$$

Thus,

$$F_Y(y) = \begin{cases} 0, & \text{if } y \le 0, \\ F_X(5) - F_X(5 - y) + F_X(20) - F_X(20 - y), & \text{if } 0 \le y \le 5, \\ F_X(20) - F_X(20 - y), & \text{if } 5 < y \le 15, \\ 1, & \text{if } y > 15. \end{cases}$$

Differentiating, we obtain

$$f_Y(y) = \begin{cases} f_X(5 - y) + f_X(20 - y), & \text{if } 0 \le y \le 5, \\ f_X(20 - y), & \text{if } 5 < y \le 15, \\ 0, & \text{otherwise,} \end{cases}$$

consistent with the result of Example 3.14.

Solution to Problem 4.5. Let $Z = |X - Y|$. We have

$$F_Z(z) = P\big(|X - Y| \le z\big) = 1 - (1 - z)^2.$$

(To see this, draw the event of interest as a subset of the unit square and calculate its area.) Taking derivatives, the desired PDF is

$$f_Z(z) = \begin{cases} 2(1 - z), & \text{if } 0 \le z \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 4.6. Let $Z = |X - Y|$. To find the CDF, we integrate the joint PDF of X and Y over the region where $|X - Y| \leq z$ for a given z. In the case where $z \leq 0$ or $z \geq 1$, the CDF is 0 and 1, respectively. In the case where $0 < z < 1$, we have

$$F_Z(z) = \mathbf{P}(X - Y \leq z,\, X \geq Y) + \mathbf{P}(Y - X \leq z,\, X < Y).$$

The events $\{X - Y \leq z,\, X \geq Y\}$ and $\{Y - X \leq z,\, X < Y\}$ can be identified with subsets of the given triangle. After some calculation using triangle geometry, the areas of these subsets can be verified to be $z/2 + z^2/4$ and $1/4 - (1-z)^2/4$, respectively. Therefore, since $f_{X,Y}(x, y) = 1$ for all (x, y) in the given triangle,

$$F_Z(z) = \left(\frac{z}{2} + \frac{z^2}{4}\right) + \left(\frac{1}{4} - \frac{(1-z)^2}{4}\right) = z.$$

Thus,

$$F_Z(z) = \begin{cases} 0, & \text{if } z \leq 0, \\ z, & \text{if } 0 < z < 1, \\ 1, & \text{if } z \geq 1. \end{cases}$$

By taking the derivative with respect to z, we obtain

$$f_Z(z) = \begin{cases} 1, & \text{if } 0 \leq z \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 4.7. Let X and Y be the two points, and let $Z = \max\{X, Y\}$. For any $t \in [0, 1]$, we have

$$\mathbf{P}(Z \leq t) = \mathbf{P}(X \leq t)\mathbf{P}(Y \leq t) = t^2,$$

and by differentiating, the corresponding PDF is

$$f_Z(z) = \begin{cases} 0, & \text{if } z \leq 0, \\ 2z, & \text{if } 0 \leq z \leq 1, \\ 0, & \text{if } z \geq 1. \end{cases}$$

Thus, we have

$$\mathbf{E}[Z] = \int_{-\infty}^{\infty} z f_Z(z) dz = \int_0^1 2z^2 dz = \frac{2}{3}.$$

The distance of the largest of the two points to the right endpoint is $1 - Z$, and its expected value is $1 - \mathbf{E}[Z] = 1/3$. A symmetric argument shows that the distance of the smallest of the two points to the left endpoint is also 1/3. Therefore, the expected distance between the two points must also be 1/3.

Solution to Problem 4.8. Note that $f_X(x)$ and $f_Y(z - x)$ are nonzero only when $x \geq 0$ and $x \leq z$, respectively. Thus, in the convolution formula, we only need to integrate for x ranging from 0 to z:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx = \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)}\, dx = \lambda^2 e^{-z} \int_0^z dx = \lambda^2 z e^{-\lambda z}.$$

in agreement with the earlier answer. The solution for the case $z < 0$ is obtained with a similar calculation.

Solution to Problem 4.10. We first note that the range of possible values of Z are the integers from the range $[1, 5]$. Thus we have

$$p_Z(z) = 0, \quad \text{if } z \neq 1, 2, 3, 4, 5.$$

We calculate $p_Z(z)$ for each of the values $z = 1, 2, 3, 4, 5$, using the convolution formula. We have

$$p_Z(1) = \sum_x p_X(x) p_Y(1 - x) = p_X(1) p_Y(0) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6},$$

where the second equality above is based on the fact that for $x \neq 1$ either $p_X(x)$ or $p_Y(1 - x)$ (or both) is zero. Similarly, we obtain

$$p_Z(2) = p_X(1)p_Y(1) + p_X(2)p_Y(0) = \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{18},$$

$$p_Z(3) = p_X(1)p_Y(2) + p_X(2)p_Y(1) + p_X(3)p_Y(0) = \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{3},$$

$$p_Z(4) = p_X(2)p_Y(2) + p_X(3)p_Y(1) = \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{6},$$

$$p_Z(5) = p_X(3)p_Y(2) = \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}.$$

Solution to Problem 4.11. The convolution of two Poisson PMFs is of the form

$$\sum_{i=0}^{k} \frac{\lambda^i e^{-\lambda}}{i!} \cdot \frac{\mu^{k-i} e^{-\mu}}{(k-i)!} = e^{-(\lambda+\mu)} \sum_{i=0}^{k} \frac{\lambda^i \mu^{k-i}}{i!\,(k-i)!}.$$

We have

$$(\lambda + \mu)^k = \sum_{i=0}^{k} \binom{k}{i} \lambda^i \mu^{k-i} = \sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!} \lambda^i \mu^{k-i}.$$

Thus, the desired PMF is

$$\frac{e^{-(\lambda+\mu)}}{k!} \sum_{i=0}^{k} \frac{k!\,\lambda^i \mu^{k-i}}{i!\,(k-i)!} = \frac{e^{-(\lambda+\mu)}}{k!} (\lambda + \mu)^k,$$

which is a Poisson PMF with mean $\lambda + \mu$.

Solution to Problem 4.12. Let $V = X + Y$. As in Example 4.10, the PDF of V is

$$f_V(v) = \begin{cases} v, & 0 \leq v \leq 1, \\ 2 - v, & 1 \leq v \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Let $W = X + Y + Z = V + Z$. We convolve the PDFs f_V and f_Z, to obtain

$$f_W(w) = \int f_V(v) f_Z(w - v)\, dv.$$

We first need to determine the limits of the integration. Since $f_V(v) = 0$ outside the range $0 \le v \le 2$, and $f_W(w - v) = 0$ outside the range $0 \le w - v \le 1$, we see that the integrand can be nonzero only if

$$0 \le v \le 2, \qquad \text{and} \qquad w - 1 \le v \le w.$$

We consider three separate cases. If $w \le 1$, we have

$$f_W(w) = \int_0^w f_V(v) f_Z(w - v)\, dv = \int_0^w v\, dv = \frac{w^2}{2}.$$

If $1 \le w \le 2$, we have

$$\begin{aligned} f_W(w) &= \int_{w-1}^{w} f_V(v) f_Z(w - v)\, dv \\ &= \int_{w-1}^{1} v\, dv + \int_1^w (2 - v)\, dv \\ &= \frac{1}{2} - \frac{(w-1)^2}{2} - \frac{(w-2)^2}{2} + \frac{1}{2}. \end{aligned}$$

Finally, if $2 \le w \le 3$, we have

$$f_W(w) = \int_{w-1}^{2} f_V(v) f_Z(w - v)\, dv = \int_{w-1}^{2} (2 - v)\, dv = \frac{(3-w)^2}{2}.$$

To summarize,

$$f_W(w) = \begin{cases} w^2/2, & 0 \le w \le 1, \\ 1 - (w-1)^2/2 - (2-w)^2/2, & 1 \le w \le 2, \\ (3-w)^2/2, & 2 \le w \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 4.13. We have $X - Y = X + Z - (a + b)$, where $Z = a + b - Y$ is distributed identically with X and Y. Thus, the PDF of $X + Z$ is the same as the PDF of $X + Y$, and the PDF of $X - Y$ is obtained by shifting the PDF of $X + Y$ to the left by $a + b$.

Solution to Problem 4.14. For all $z \ge 0$, we have, using the independence of X and Y, and the form of the exponential CDF,

$$\begin{aligned} F_Z(z) &= \mathbf{P}\big(\min\{X, Y\} \le z\big) \\ &= 1 - \mathbf{P}\big(\min\{X, Y\} > z\big) \\ &= 1 - \mathbf{P}(X > z,\, Y > z) \\ &= 1 - \mathbf{P}(X > z)\mathbf{P}(Y > z) \\ &= 1 - e^{-\lambda z} e^{-\mu z} \\ &= 1 - e^{-(\lambda + \mu) z}. \end{aligned}$$

This is recognized as the exponential CDF with parameter $\lambda + \mu$. Thus, the minimum of two independent exponentials with parameters λ and μ is an exponential with parameter $\lambda + \mu$.

Solution to Problem 4.17. Because the covariance remains unchanged when we add a constant to a random variable, we can assume without loss of generality that X and Y have zero mean. We then have

$$\mathrm{cov}(X - Y, X + Y) = \mathbf{E}\big[(X - Y)(X + Y)\big] = \mathbf{E}[X^2] - \mathbf{E}[Y^2] = \mathrm{var}(X) - \mathrm{var}(Y) = 0,$$

since X and Y were assumed to have the same variance.

Solution to Problem 4.18. We have

$$\mathrm{cov}(R, S) = \mathbf{E}[RS] - \mathbf{E}[R]\mathbf{E}[S] = \mathbf{E}[WX + WY + X^2 + XY] = \mathbf{E}[X^2] = 1,$$

and

$$\mathrm{var}(R) = \mathrm{var}(S) = 2,$$

so

$$\rho(R, S) = \frac{\mathrm{cov}(R, S)}{\sqrt{\mathrm{var}(R)\mathrm{var}(S)}} = \frac{1}{2}.$$

We also have

$$\mathrm{cov}(R, T) = \mathbf{E}[RT] - \mathbf{E}[R]\mathbf{E}[T] = \mathbf{E}[WY + WZ + XY + XZ] = 0,$$

so that

$$\rho(R, T) = 0.$$

Solution to Problem 4.19. To compute the correlation coefficient

$$\rho(X, Y) = \frac{\mathrm{cov}(X, Y)}{\sigma_X \sigma_Y},$$

we first compute the covariance:

$$\begin{aligned}
\mathrm{cov}(X, Y) &= \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y] \\
&= \mathbf{E}[aX + bX^2 + cX^3] - \mathbf{E}[X]\mathbf{E}[Y] \\
&= a\mathbf{E}[X] + b\mathbf{E}[X^2] + c\mathbf{E}[X^3] \\
&= b.
\end{aligned}$$

We also have

$$\begin{aligned}
\mathrm{var}(Y) &= \mathrm{var}(a + bX + cX^2) \\
&= \mathbf{E}\big[(a + bX + cX^2)^2\big] - \big(\mathbf{E}[a + bX + cX^2]\big)^2 \\
&= (a^2 + 2ac + b^2 + 3c^2) - (a^2 + c^2 + 2ac) \\
&= b^2 + 2c^2,
\end{aligned}$$

and therefore, using the fact $\mathrm{var}(X) = 1$,

$$\rho(X, Y) = \frac{b}{\sqrt{b^2 + 2c^2}}.$$

Solution to Problem 4.22. If the gambler's fortune at the beginning of a round is a, the gambler bets $a(2p-1)$. He therefore gains $a(2p-1)$ with probability p, and loses $a(2p-1)$ with probability $1-p$. Thus, his expected fortune at the end of a round is

$$a\big(1 + p(2p-1) - (1-p)(2p-1)\big) = a\big(1 + (2p-1)^2\big).$$

Let X_k be the fortune after the kth round. Using the preceding calculation, we have

$$\mathbf{E}[X_{k+1} \,|\, X_k] = \big(1 + (2p-1)^2\big) X_k.$$

Using the law of iterated expectations, we obtain

$$\mathbf{E}[X_{k+1}] = \big(1 + (2p-1)^2\big)\mathbf{E}[X_k],$$

and

$$\mathbf{E}[X_1] = \big(1 + (2p-1)^2\big)x.$$

We conclude that

$$\mathbf{E}[X_n] = \big(1 + (2p-1)^2\big)^n x.$$

Solution to Problem 4.23. (a) Let W be the number of hours that Nat waits. We have

$$\mathbf{E}[X] = \mathbf{P}(0 \le X \le 1)\mathbf{E}[W \,|\, 0 \le X \le 1] + \mathbf{P}(X > 1)\mathbf{E}[W \,|\, X > 1].$$

Since $W > 0$ only if $X > 1$, we have

$$\mathbf{E}[W] = \mathbf{P}(X > 1)\mathbf{E}[W \,|\, X > 1] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

(b) Let D be the duration of a date. We have $\mathbf{E}[D \,|\, 0 \le X \le 1] = 3$. Furthermore, when $X > 1$, the conditional expectation of D given X is $(3 - X)/2$. Hence, using the law of iterated expectations,

$$\mathbf{E}[D \,|\, X > 1] = \mathbf{E}\Big[\mathbf{E}[D \,|\, X] \ \ \Big| \, X > 1\Big] = \mathbf{E}\left[\frac{3-X}{2} \ \Big| \ X > 1\right].$$

Therefore,

$$\begin{aligned}
\mathbf{E}[D] &= \mathbf{P}(0 \le X \le 1)\mathbf{E}[D \,|\, 0 \le X \le 1] + \mathbf{P}(X > 1)\mathbf{E}[D \,|\, X > 1] \\
&= \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot \mathbf{E}\left[\frac{3-X}{2} \ \Big| \ X > 1\right] \\
&= \frac{3}{2} + \frac{1}{2}\left(\frac{3}{2} - \frac{\mathbf{E}[X \,|\, X > 1]}{2}\right) \\
&= \frac{3}{2} + \frac{1}{2}\left(\frac{3}{2} - \frac{3/2}{2}\right) \\
&= \frac{15}{8}.
\end{aligned}$$