

and therefore, using the fact $\text{var}(X) = 1$,

$$\rho(X, Y) = \frac{b}{\sqrt{b^2 + 2c^2}}.$$

Solution to Problem 4.22. If the gambler's fortune at the beginning of a round is a , the gambler bets $a(2p - 1)$. He therefore gains $a(2p - 1)$ with probability p , and loses $a(2p - 1)$ with probability $1 - p$. Thus, his expected fortune at the end of a round is

$$a(1 + p(2p - 1) - (1 - p)(2p - 1)) = a(1 + (2p - 1)^2).$$

Let X_k be the fortune after the k th round. Using the preceding calculation, we have

$$\mathbf{E}[X_{k+1} | X_k] = (1 + (2p - 1)^2)X_k.$$

Using the law of iterated expectations, we obtain

$$\mathbf{E}[X_{k+1}] = (1 + (2p - 1)^2)\mathbf{E}[X_k],$$

and

$$\mathbf{E}[X_1] = (1 + (2p - 1)^2)x.$$

We conclude that

$$\mathbf{E}[X_n] = (1 + (2p - 1)^2)^n x.$$

Solution to Problem 4.23. (a) Let W be the number of hours that Nat waits. We have

$$\mathbf{E}[X] = \mathbf{P}(0 \leq X \leq 1)\mathbf{E}[W | 0 \leq X \leq 1] + \mathbf{P}(X > 1)\mathbf{E}[W | X > 1].$$

Since $W > 0$ only if $X > 1$, we have

$$\mathbf{E}[W] = \mathbf{P}(X > 1)\mathbf{E}[W | X > 1] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

(b) Let D be the duration of a date. We have $\mathbf{E}[D | 0 \leq X \leq 1] = 3$. Furthermore, when $X > 1$, the conditional expectation of D given X is $(3 - X)/2$. Hence, using the law of iterated expectations,

$$\mathbf{E}[D | X > 1] = \mathbf{E}[\mathbf{E}[D | X] | X > 1] = \mathbf{E}\left[\frac{3 - X}{2} \mid X > 1\right].$$

Therefore,

$$\begin{aligned} \mathbf{E}[D] &= \mathbf{P}(0 \leq X \leq 1)\mathbf{E}[D | 0 \leq X \leq 1] + \mathbf{P}(X > 1)\mathbf{E}[D | X > 1] \\ &= \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot \mathbf{E}\left[\frac{3 - X}{2} \mid X > 1\right] \\ &= \frac{3}{2} + \frac{1}{2} \left(\frac{3}{2} - \frac{\mathbf{E}[X | X > 1]}{2}\right) \\ &= \frac{3}{2} + \frac{1}{2} \left(\frac{3}{2} - \frac{3/2}{2}\right) \\ &= \frac{15}{8}. \end{aligned}$$

(c) The probability that Pat will be late by more than 45 minutes is $1/8$. The number of dates before breaking up is the sum of two geometrically distributed random variables with parameter $1/8$, and its expected value is $2 \cdot 8 = 16$.

Solution to Problem 4.24. (a) Consider the following two random variables:

X = amount of time the professor devotes to his task [exponentially distributed with parameter $\lambda(y) = 1/(5 - y)$];

Y = length of time between 9 a.m. and his arrival (uniformly distributed between 0 and 4).

Note that $\mathbf{E}[Y] = 2$. We have

$$\mathbf{E}[X | Y = y] = \frac{1}{\lambda(y)} = 5 - y,$$

which implies that

$$\mathbf{E}[X | Y] = 5 - Y,$$

and

$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X | Y]] = \mathbf{E}[5 - Y] = 5 - \mathbf{E}[Y] = 5 - 2 = 3.$$

(b) Let Z be the length of time from 9 a.m. until the professor completes the task. Then,

$$Z = X + Y.$$

We already know from part (a) that $\mathbf{E}[X] = 3$ and $\mathbf{E}[Y] = 2$, so that

$$\mathbf{E}[Z] = \mathbf{E}[X] + \mathbf{E}[Y] = 3 + 2 = 5.$$

Thus the expected time that the professor leaves his office is 5 hours after 9 a.m.

(c) We define the following random variables:

W = length of time between 9 a.m. and arrival of the Ph.D. student (uniformly distributed between 9 a.m. and 5 p.m.).

R = amount of time the student will spend with the professor, if he finds the professor (uniformly distributed between 0 and 1 hour).

T = amount of time the professor will spend with the student.

Let also F be the event that the student finds the professor.

To find $\mathbf{E}[T]$, we write

$$\mathbf{E}[T] = \mathbf{P}(F)\mathbf{E}[T | F] + \mathbf{P}(F^c)\mathbf{E}[T | F^c]$$

Using the problem data,

$$\mathbf{E}[T | F] = \mathbf{E}[R] = \frac{1}{2}$$

(this is the expected value of a uniform distribution ranging from 0 to 1),

$$\mathbf{E}[T | F^c] = 0$$

(since the student leaves if he does not find the professor). We have

$$\mathbf{E}[T] = \mathbf{E}[T | F] \mathbf{P}(F) = \frac{1}{2} \mathbf{P}(F),$$

so we need to find $\mathbf{P}(F)$.

In order for the student to find the professor, his arrival should be between the arrival and the departure of the professor. Thus

$$\mathbf{P}(F) = \mathbf{P}(Y \leq W \leq X + Y).$$

We have that W can be between 0 (9 a.m.) and 8 (5 p.m.), but $X + Y$ can be any value greater than 0. In particular, it may happen that the sum is greater than the upper bound for W . We write

$$\mathbf{P}(F) = \mathbf{P}(Y \leq W \leq X + Y) = 1 - (\mathbf{P}(W < Y) + \mathbf{P}(W > X + Y))$$

We have

$$\mathbf{P}(W < Y) = \int_0^4 \frac{1}{4} \int_0^y \frac{1}{8} dw dy = \frac{1}{4}$$

and

$$\begin{aligned} \mathbf{P}(W > X + Y) &= \int_0^4 \mathbf{P}(W > X + Y | Y = y) f_Y(y) dy \\ &= \int_0^4 \mathbf{P}(X < W - Y | Y = y) f_Y(y) dy \\ &= \int_0^4 \int_y^8 F_{X|Y}(w - y) f_W(w) f_Y(y) dw dy \\ &= \int_0^4 \frac{1}{4} \int_y^8 \frac{1}{8} \int_0^{w-y} \frac{1}{5-y} e^{-\frac{x}{5-y}} dx dw dy \\ &= \frac{12}{32} + \frac{1}{32} \int_0^4 (5-y) e^{-\frac{8-y}{5-y}} dy. \end{aligned}$$

Integrating numerically, we have

$$\int_0^4 (5-y) e^{-\frac{8-y}{5-y}} dy = 1.7584.$$

Thus,

$$\mathbf{P}(Y \leq W \leq X + Y) = 1 - (\mathbf{P}(W < Y) + \mathbf{P}(W > X + Y)) = 1 - 0.68 = 0.32.$$

The expected amount of time the professor will spend with the student is then

$$\mathbf{E}[T] = \frac{1}{2} \mathbf{P}(F) = \frac{1}{2} 0.32 = 0.16 = 9.6 \text{ mins.}$$

Next, we want to find the expected time the professor will leave his office. Let Z be the length of time measured from 9 a.m. until he leaves his office. If the professor

doesn't spend any time with the student, then Z will be equal to $X + Y$. On the other hand, if the professor is interrupted by the student, then the length of time will be equal to $X + Y + R$. This is because the professor will spend the same amount of total time on the task regardless of whether he is interrupted by the student. Therefore,

$$\mathbf{E}[Z] = \mathbf{P}(F)\mathbf{E}[Z | F] + \mathbf{P}(F^c)\mathbf{E}[Z | F^c] = \mathbf{P}(F)\mathbf{E}[X + Y + R] + \mathbf{P}(F^c)\mathbf{E}[X + Y].$$

Using the results of the earlier calculations,

$$\mathbf{E}[X + Y] = 5,$$

$$\mathbf{E}[X + Y + R] = \mathbf{E}[X + Y] + \mathbf{E}[R] = 5 + \frac{1}{2} = \frac{11}{2}.$$

Therefore,

$$\mathbf{E}[Z] = 0.68 \cdot 5 + 0.32 \cdot \frac{11}{2} = 5.16.$$

Thus the expected time the professor will leave his office is 5.16 hours after 9 a.m.

Solution to Problem 4.29. The transform is given by

$$M(s) = \mathbf{E}[e^{sX}] = \frac{1}{2}e^s + \frac{1}{4}e^{2s} + \frac{1}{4}e^{3s}.$$

We have

$$\mathbf{E}[X] = \left. \frac{d}{ds} M(s) \right|_{s=0} = \frac{1}{2} + \frac{2}{4} + \frac{3}{4} = \frac{7}{4},$$

$$\mathbf{E}[X^2] = \left. \frac{d^2}{ds^2} M(s) \right|_{s=0} = \frac{1}{2} + \frac{4}{4} + \frac{9}{4} = \frac{15}{4},$$

$$\mathbf{E}[X^3] = \left. \frac{d^3}{ds^3} M(s) \right|_{s=0} = \frac{1}{2} + \frac{8}{4} + \frac{27}{4} = \frac{37}{4}.$$

Solution to Problem 4.30. The transform associated with X is

$$M_X(s) = e^{s^2/2}.$$

By taking derivatives with respect to s , we find that

$$\mathbf{E}[X] = 0, \quad \mathbf{E}[X^2] = 1, \quad \mathbf{E}[X^3] = 0, \quad \mathbf{E}[X^4] = 3.$$

Solution to Problem 4.31. The transform is

$$M(s) = \frac{\lambda}{\lambda - s}.$$

Thus,

$$\frac{d}{ds} M(s) = \frac{\lambda}{(\lambda - s)^2}, \quad \frac{d^2}{ds^2} M(s) = \frac{2\lambda}{(\lambda - s)^3}, \quad \frac{d^3}{ds^3} M(s) = \frac{6\lambda}{(\lambda - s)^4},$$

$$\frac{d^4}{ds^4}M(s) = \frac{24\lambda}{(\lambda - s)^5}, \quad \frac{d^5}{ds^5}M(s) = \frac{120\lambda}{(\lambda - s)^6}.$$

By setting $s = 0$, we obtain

$$\mathbf{E}[X^3] = \frac{6}{\lambda^3}, \quad \mathbf{E}[X^4] = \frac{24}{\lambda^4}, \quad \mathbf{E}[X^5] = \frac{120}{\lambda^5}.$$

Solution to Problem 4.32. (a) We must have $M(0) = 1$. Only the first option satisfies this requirement.

(b) We have

$$\mathbf{P}(X = 0) = \lim_{s \rightarrow -\infty} M(s) = e^{2(e^{-1}-1)} \approx 0.2825.$$

Solution to Problem 4.33. We recognize this transform as corresponding to the following mixture of exponential PDFs:

$$f_X(x) = \begin{cases} \frac{1}{3} \cdot 2e^{-2x} + \frac{2}{3} \cdot 3e^{-3x}, & \text{for } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

By the inversion theorem, this must be the desired PDF.

Solution to Problem 4.34. For $i = 1, 2, 3$, let X_i , $i = 1, 2, 3$, be a Bernoulli random variable that takes the value 1 if the i th player is successful. We have $X = X_1 + X_2 + X_3$. Let $q_i = 1 - p_i$. Convolution of the PMFs of X_1 and X_2 yields the PMF of $Z = X_1 + X_2$:

$$p_Z(z) = \begin{cases} q_1 q_2, & \text{if } z = 0, \\ q_1 p_2 + p_1 q_2, & \text{if } z = 1, \\ p_1 p_2, & \text{if } z = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Convolution of the PMFs of Z and X_3 yields the PMF of $X = X_1 + X_2 + X_3$:

$$p_X(x) = \begin{cases} q_1 q_2 q_3, & \text{if } x = 0, \\ p_1 q_2 q_3 + q_1 p_2 q_3 + q_1 q_2 p_3, & \text{if } x = 1, \\ q_1 p_2 p_3 + p_1 q_2 p_3 + p_1 p_2 q_3, & \text{if } x = 2, \\ p_1 p_2 p_3, & \text{if } x = 3, \\ 0, & \text{otherwise.} \end{cases}$$

The transform associated with X is the product of the transforms associated with X_i , $i = 1, 2, 3$. We have

$$M_X(s) = (q_1 + p_1 e^s)(q_2 + p_2 e^s)(q_3 + p_3 e^s).$$

By carrying out the multiplications above, and by examining the coefficients of the terms e^{ks} , we obtain the probabilities $\mathbf{P}(X = k)$. These probabilities are seen to coincide with the ones computed by convolution.

Solution to Problem 4.35. We first find c by using the equation

$$1 = M_X(0) = c \cdot \frac{3 + 4 + 2}{3 - 1},$$

so that $c = 2/9$. We then obtain

$$\mathbf{E}[X] = \left. \frac{dM_X}{ds}(s) \right|_{s=0} = \frac{2}{9} \cdot \left. \frac{(3 - e^s)(8e^{2s} + 6e^{3s}) + e^s(3 + 4e^{2s} + 2e^{3s})}{(3 - e^s)^2} \right|_{s=0} = \frac{37}{18}.$$

We now use the identity

$$\frac{1}{3 - e^s} = \frac{1}{3} \cdot \frac{1}{1 - e^s/3} = \frac{1}{3} \left(1 + \frac{e^s}{3} + \frac{e^{2s}}{9} + \cdots \right),$$

which is valid as long as s is small enough so that $e^s < 3$. It follows that

$$M_X(s) = \frac{2}{9} \cdot \frac{1}{3} \cdot (3 + 4e^{2s} + 2e^{3s}) \cdot \left(1 + \frac{e^s}{3} + \frac{e^{2s}}{9} + \cdots \right).$$

By identifying the coefficients of e^{0s} and e^s , we obtain

$$p_X(0) = \frac{2}{9}, \quad p_X(1) = \frac{2}{27}.$$

Let $A = \{X \neq 0\}$. We have

$$p_{X|\{X \in A\}}(k) = \begin{cases} \frac{p_X(k)}{\mathbf{P}(A)}, & \text{if } k \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

so that

$$\begin{aligned} \mathbf{E}[X | X \neq 0] &= \sum_{k=1}^{\infty} k p_{X|A}(k) \\ &= \sum_{k=1}^{\infty} k \frac{p_X(k)}{\mathbf{P}(A)} \\ &= \frac{\mathbf{E}[X]}{1 - p_X(0)} \\ &= \frac{37/18}{7/9} \\ &= \frac{37}{14}. \end{aligned}$$

Solution to Problem 4.36. (a) We have $U = X$ if $X = 1$, which happens with probability $1/3$, and $U = Z$ if $X = 0$, which happens with probability $2/3$. Therefore, U is a mixture of random variables and the associated transform is

$$M_U(s) = \mathbf{P}(X = 1)M_Y(s) + \mathbf{P}(X = 0)M_Z(s) = \frac{1}{3} \cdot \frac{2}{2 - s} + \frac{2}{3}e^{3(e^s - 1)}.$$

(b) Let $V = 2Z + 3$. We have

$$M_V(s) = e^{3s} M_Z(2s) = e^{3s} e^{3(e^{2s}-1)} = e^{3(s-1+e^{2s})}.$$

(c) Let $W = Y + Z$. We have

$$M_W(s) = M_Y(s)M_Z(s) = \frac{2}{2-s} e^{3(e^s-1)}.$$

Solution to Problem 4.37. Let X be the number of different types of pizza ordered. Let X_i be the random variable defined by

$$X_i = \begin{cases} 1, & \text{if a type } i \text{ pizza is ordered by at least one customer,} \\ 0, & \text{otherwise.} \end{cases}$$

We have $X = X_1 + \cdots + X_n$, and by the law of iterated expectations,

$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X | K]] = \mathbf{E}[\mathbf{E}[X_1 + \cdots + X_n | K]] = n \mathbf{E}[\mathbf{E}[X_1 | K]].$$

Furthermore, since the probability that a customer does not order a pizza of type 1 is $(n-1)/n$, we have

$$\mathbf{E}[X_1 | K = k] = 1 - \left(\frac{n-1}{n}\right)^k,$$

so that

$$\mathbf{E}[X_1 | K] = 1 - \left(\frac{n-1}{n}\right)^K.$$

Thus, denoting

$$p = \frac{n-1}{n},$$

we have

$$\mathbf{E}[X] = n \mathbf{E}[1 - p^K] = n - n \mathbf{E}[p^K] = n - n \mathbf{E}[e^{K \log p}] = n - n M_K(\log p).$$

Solution to Problem 4.41. (a) Let N be the number of people that enter the elevator. The corresponding transform is $M_N(s) = e^{\lambda(e^s-1)}$. Let $M_X(s)$ be the common transform associated with the random variables X_i . Since X_i is uniformly distributed within $[0, 1]$, we have

$$M_X(s) = \frac{e^s - 1}{s}.$$

The transform $M_Y(s)$ is found by starting with the transform $M_N(s)$ and replacing each occurrence of e^s with $M_X(s)$. Thus,

$$M_Y(s) = e^{\lambda(M_X(s)-1)} = e^{\lambda\left(\frac{e^s-1}{s}-1\right)}.$$

(b) We have using the chain rule

$$\mathbf{E}[Y] = \left. \frac{d}{ds} M_Y(s) \right|_{s=0} = \left. \frac{d}{ds} M_X(s) \right|_{s=0} \cdot \left. \lambda e^{\lambda(M_X(s)-1)} \right|_{s=0} = \frac{1}{2} \cdot \lambda = \frac{\lambda}{2},$$

where we have used the fact that $M_X(0) = 1$.

(c) From the law of iterated expectations we obtain

$$\mathbf{E}[Y] = \mathbf{E}[\mathbf{E}[Y | N]] = \mathbf{E}[N\mathbf{E}[X]] = \mathbf{E}[N]\mathbf{E}[X] = \frac{\lambda}{2}.$$

Solution to Problem 4.42. Take X and Y to be normal with means 1 and 2, respectively, and very small variances. Consider the random variable that takes the value of X with some probability p and the value of Y with probability $1 - p$. This random variable takes values near 1 and 2 with relatively high probability, but takes values near its mean (which is $3 - 2p$) with relatively low probability. Thus, this random variable is not normal.

Now let N be a random variable taking only the values 1 and 2 with probabilities p and $1 - p$, respectively. The sum of a number N of independent normal random variables with mean equal to 1 and very small variance is a mixture of the type discussed above, which is not normal.

Solution to Problem 4.43. (a) Using the total probability theorem, we have

$$\mathbf{P}(X > 4) = \sum_{k=0}^4 \mathbf{P}(k \text{ lights are red}) \mathbf{P}(X > 4 | k \text{ lights are red}).$$

We have

$$\mathbf{P}(k \text{ lights are red}) = \binom{4}{k} \left(\frac{1}{2}\right)^4.$$

The conditional PDF of X given that k lights are red, is normal with mean k minutes and standard deviation $(1/2)\sqrt{k}$. Thus, X is a mixture of normal random variables and the transform associated with its (unconditional) PDF is the corresponding mixture of the transforms associated with the (conditional) normal PDFs. However, X is not normal, because a mixture of normal PDFs need not be normal. The probability $\mathbf{P}(X > 4 | k \text{ lights are red})$ can be computed from the normal tables for each k , and $\mathbf{P}(X > 4)$ is obtained by substituting the results in the total probability formula above.

(b) Let K be the number of traffic lights that are found to be red. We can view X as the sum of K independent normal random variables. Thus the transform associated with X can be found by replacing in the binomial transform $M_K(s) = (1/2 + (1/2)e^s)^4$ the occurrence of e^s by the normal transform corresponding to $\mu = 1$ and $\sigma = 1/2$. Thus

$$M_X(s) = \left(\frac{1}{2} + \frac{1}{2} \left(e^{\frac{(1/2)^2 s^2}{2} + s} \right) \right)^4.$$

Note that by using the formula for the transform, we cannot easily obtain the probability $\mathbf{P}(X > 4)$.