# CHAPTER 5

**Solution to Problem 5.1.** (a) We have $\sigma_{M_n} = 1/\sqrt{n}$, so in order that $\sigma_{M_n} \leq 0.01$, we must have $n \geq 10,000$.

(b) We want to have

$$\mathbf{P}\Big(|M_n - h| \leq 0.05\Big) \geq 0.99.$$

Using the facts $h = \mathbf{E}[M_n]$, $\sigma^2_{M_n} = 1/n$, and the Chebyshev inequality, we have

$$\begin{aligned}\mathbf{P}\Big(|M_n - h| \leq 0.05\Big) &= \mathbf{P}\Big(|M_n - \mathbf{E}[M_n]| \leq 0.05\Big)\\ &= 1 - \mathbf{P}\Big(|M_n - \mathbf{E}[M_n]| \geq 0.05\Big)\\ &\geq 1 - \frac{1/n}{(0.05)^2}.\end{aligned}$$

Thus, we must have

$$1 - \frac{1/n}{(0.05)^2} \geq 0.99,$$

which yields $n \geq 40,000$.

(c) Based on Example 5.3, $\sigma^2_{X_i} \leq (0.6)^2/4$, so he should use 0.3 meters in place of 1.0 meters as the estimate of the standard deviation of the samples $X_i$ in the calculations of parts (a) and (b). In the case of part (a), we have $\sigma_{M_n} = 0.3/\sqrt{n}$, so in order that $\sigma_{M_n} \leq 0.01$, we must have $n \geq 900$. In the case of part (b), we have $\sigma_{M_n} = 0.3/\sqrt{n}$, so in order that $\sigma_{M_n} \leq 0.01$, we must have $n \geq 900$. In the case of part (a), we must have

$$1 - \frac{0.09/n}{(0.05)^2} \geq 0.99,$$

which yields $n \geq 3,600$.

**Solution to Problem 5.4.** Proceeding as in Example 5.5, the best guarantee that can be obtained from the Chebyshev inequality is

$$\mathbf{P}\Big(|M_n - f| \geq \epsilon\Big) \leq \frac{1}{4n\epsilon^2}.$$

(a) If $\epsilon$ is reduced to half its original value, and in order to keep the bound $1/(4n\epsilon^2)$ constant, the sample size $n$ must be made four times larger.

(b) If the error probability $\delta$ is to be reduced to $\delta/2$, while keeping $\epsilon$ the same, the sample size has to be doubled.

**Solution to Problem 5.5.** In cases (a), (b), and (c), we show that $Y_n$ converges to 0 in probability. In case (d), we show that $Y_n$ converges to 1 in probability.

(a) For any $\epsilon > 0$, we have

$$\mathbf{P}\Big(|Y_n| \geq \epsilon\Big) = 0,$$

for all $n$ with $1/n < \epsilon$, so $\mathbf{P}\big(|Y_n| \geq \epsilon\big) \to 0$.

(b) For all $\epsilon \in (0, 1)$, we have

$$\mathbf{P}\big(|Y_n| \geq \epsilon\big) = \mathbf{P}\big(|X_n|^n \geq \epsilon\big) = \mathbf{P}\big(X_n \geq \epsilon^{1/n}\big) + \mathbf{P}\big(X_n \leq -\epsilon^{1/n}\big) = 1 - \epsilon^{1/n},$$

and the two terms in the right-hand side converge to 0, since $\epsilon^{1/n} \to 1$.

(c) Since $X_1, X_2, \ldots$ are independent random variables, we have

$$\mathbf{E}[Y_n] = \mathbf{E}[X_1] \cdots \mathbf{E}[X_n] = 0.$$

Also

$$\mathrm{var}(Y_n) = \mathbf{E}[Y_n^2] = \mathbf{E}[X_1^2] \cdots \mathbf{E}[X_n^2] = \mathrm{var}(X_1)^n = \left(\frac{4}{12}\right)^n,$$

so $\mathrm{var}(Y_n) \to 0$. Since all $Y_n$ have 0 as a common mean, from Chebyshev's inequality it follows that $Y_n$ converges to 0 in probability.

(d) We have for all $\epsilon \in (0, 1)$, using the independence of $X_1, X_2, \ldots$,

$$\begin{aligned}
\mathbf{P}\big(|Y_n - 1| \geq \epsilon\big) &= \mathbf{P}\big(\max\{X_1, \ldots, X_n\} \geq 1 + \epsilon\big) + \mathbf{P}\big(\max\{X_1, \ldots, X_n\} \leq 1 - \epsilon\big) \\
&= \mathbf{P}(X_1 \leq 1 - \epsilon, \ldots, X_n \leq 1 - \epsilon) \\
&= \big(\mathbf{P}(X_1 \leq 1 - \epsilon)\big)^n \\
&= \left(1 - \frac{\epsilon}{2}\right)^n.
\end{aligned}$$

Hence $\mathbf{P}\big(|Y_n - 1| \geq \epsilon\big) \to 0$.

**Solution to Problem 5.8.** Let $S$ be the number of times that the result was odd, which is a binomial random variable, with parameters $n = 100$ and $p = 0.5$, so that $\mathbf{E}[X] = 100 \cdot 0.5 = 50$ and $\sigma_S = \sqrt{100 \cdot 0.5 \cdot 0.5} = \sqrt{25} = 5$. Using the normal approximation to the binomial, we find

$$\mathbf{P}(S > 55) = \mathbf{P}\left(\frac{S - 50}{5} > \frac{55 - 50}{5}\right) \approx 1 - \Phi(1) = 1 - 0.8413 = 0.1587.$$

A better approximation can be obtained by using the de Moivre-Laplace approximation, which yields

$$\begin{aligned}
\mathbf{P}(S > 55) = \mathbf{P}(S \geq 55.5) &= \mathbf{P}\left(\frac{S - 50}{5} > \frac{55.5 - 50}{5}\right) \\
&\approx 1 - \Phi(1.1) = 1 - 0.8643 = 0.1357.
\end{aligned}$$

**Solution to Problem 5.9.** (a) Let $S$ be the number of crash-free days, which is a binomial random variable with parameters $n = 50$ and $p = 0.95$, so that $\mathbf{E}[X] = 50 \cdot 0.95 = 47.5$ and $\sigma_S = \sqrt{50 \cdot 0.95 \cdot 0.05} = 1.54$. Using the normal approximation to the binomial, we find

$$\mathbf{P}(S \geq 45) = \mathbf{P}\left(\frac{S - 47.5}{1.54} \geq \frac{45 - 47.5}{1.54}\right) \approx 1 - \Phi(-1.62) = \Phi(1.62) = 0.9474.$$

A better approximation can be obtained by using the de Moivre-Laplace approximation, which yields

$$\mathbf{P}(S \geq 45) = \mathbf{P}(S > 44.5) = \mathbf{P}\left(\frac{S - 47.5}{1.54} \geq \frac{44.5 - 47.5}{1.54}\right)$$
$$\approx 1 - \Phi(-1.95) = \Phi(1.95) = 0.9744.$$

(b) The random variable $S$ is binomial with parameter $p = 0.95$. However, the random variable $50 - S$ (the number of crashes) is also binomial with parameter $p = 0.05$. Since the Poisson approximation is exact in the limit of small $p$ and large $n$, it will give more accurate results if applied to $50 - S$. We will therefore approximate $50 - S$ by a Poisson random variable with parameter $\lambda = 50 \cdot 0.05 = 2.5$. Thus,

$$\begin{aligned}\mathbf{P}(S \geq 45) &= \mathbf{P}(50 - S \leq 5)\\ &= \sum_{k=0}^{5} \mathbf{P}(n - S = k)\\ &= \sum_{k=0}^{5} e^{-\lambda} \frac{\lambda^k}{k!}\\ &= 0.958.\end{aligned}$$

It is instructive to compare with the exact probability which is

$$\sum_{k=0}^{5} \binom{50}{k} 0.05^k \cdot 0.95^{50-k} = 0.962.$$

Thus, the Poisson approximation is closer. This is consistent with the intuition that the normal approximation to the binomial works well when $p$ is close to 0.5 or $n$ is very large, which is not the case here. On the other hand, the calculations based on the normal approximation are generally less tedious.

**Solution to Problem 5.10.** (a) Let $S_n = X_1 + \cdots + X_n$ be the total number of gadgets produced in $n$ days. Note that the mean, variance, and standard deviation of $S_n$ is $5n$, $9n$, and $3\sqrt{n}$, respectively. Thus,

$$\begin{aligned}\mathbf{P}(S_{100} < 440) &= \mathbf{P}(S_{100} \leq 439.5)\\ &= \mathbf{P}\left(\frac{S_{100} - 500}{30} < \frac{439.5 - 500}{30}\right)\\ &\approx \Phi(\frac{439.5 - 500}{30})\\ &= \Phi(-2.02)\\ &= 1 - \Phi(2.02)\\ &= 1 - 0.9783\\ &= 0.0217.\end{aligned}$$

(b) The requirement $\mathbf{P}(S_n \geq 200 + 5n) \leq 0.05$ translates to

$$\mathbf{P}\left(\frac{S_n - 5n}{3\sqrt{n}} \geq \frac{200}{3\sqrt{n}}\right) \leq 0.05,$$

or, using a normal approximation,

$$1 - \Phi\left(\frac{200}{3\sqrt{n}}\right) \leq 0.05,$$

and

$$\Phi\left(\frac{200}{3\sqrt{n}}\right) \geq 0.95.$$

From the normal tables, we obtain $\Phi(1.65) \approx 0.95$, and therefore,

$$\frac{200}{3\sqrt{n}} \geq 1.65,$$

which finally yields $n \leq 1632$.

(c) The event $N \geq 220$ (it takes at least 220 days to exceed 1000 gadgets) is the same as the event $S_{219} \leq 1000$ (no more than 1000 gadgets produced in the first 219 days). Thus,

$$\begin{aligned}
\mathbf{P}(N \geq 220) &= \mathbf{P}(S_{219} \leq 1000) \\
&= \mathbf{P}\left(\frac{S_{219} - 5 \cdot 219}{3\sqrt{219}} \leq \frac{1000 - 5 \cdot 219}{3\sqrt{219}}\right) \\
&= 1 - \Phi(2.14) \\
&= 1 - 0.9838 \\
&= 0.0162.
\end{aligned}$$

**Solution to Problem 5.11.** Note that $W$ is the sample mean of 16 independent identically distributed random variables of the form $X_i - Y_i$, and a normal approximation is appropriate. The random variables $X_i - Y_i$ have zero mean, and variance equal to $2/12$. Therefore, the mean of $W$ is zero, and its variance is $(2/12)/16 = 1/96$. Thus,

$$\mathbf{P}\big(|W| < 0.001\big) = \mathbf{P}\left(\frac{|W|}{\sqrt{1/96}} < \frac{0.001}{\sqrt{1/96}}\right) \approx \Phi\big(0.001\sqrt{96}\big) - \Phi\big(-0.001\sqrt{96}\big)$$

$$= 2\Phi\big(0.001\sqrt{96}\big) - 1 = 2\Phi(0.0098) - 1 \approx 2 \cdot 0.504 - 1 = 0.008.$$

Let us also point out a somewhat different approach that bypasses the need for the normal table. Let $Z$ be a normal random variable with zero mean and standard deviation equal to $1/\sqrt{96}$. The standard deviation of $Z$, which is about 0.1, is much larger than 0.001. Thus, within the interval $[-0.001, 0.001]$, the PDF of $Z$ is approximately constant. Using the formula $\mathbf{P}(z - \delta \leq Z \leq z + \delta) \approx f_Z(z) \cdot 2\delta$, with $z = 0$ and $\delta = 0.001$, we obtain

$$\mathbf{P}\big(|W| < 0.001\big) \approx \mathbf{P}(-0.001 \leq Z \leq 0.001) \approx f_Z(0) \cdot 0.002 = \frac{0.002}{\sqrt{2\pi}(1/\sqrt{96})} = 0.0078.$$

Now $Y_n = (1/n)Y$. Thus, by Eq. (4.168), the characteristic function of $Y_n$ is

$$\Psi_{Y_n}(\omega) = \Psi_Y\left(\frac{\omega}{n}\right) = e^{-na|\omega/n|} = e^{-a|\omega|} \tag{4.177}$$

(*b*) Equation (4.177) indicates that $Y_n$ is also a Cauchy r.v. with parameter $a$, and its pdf is the same as that of $X_i$.

(*c*) Since the characteristic function of $Y_n$ is independent of $n$ and so is its pdf, $Y_n$ does not tend to a normal r.v. as $n \to \infty$, and so the central limit theorem does not hold in this case.

**4.85.** Let $Y$ be a binomial r.v. with parameters $(n, p)$. Using the central limit theorem, derive the approximation formula

$$P(Y \le y) \approx \Phi\left(\frac{y - np}{\sqrt{np(1-p)}}\right) \tag{4.178}$$

where $\Phi(z)$ is the cdf of a standard normal r.v. [Eq. (2.73)].

We saw in Prob. 4.54 that if $X_1, \ldots, X_n$ are independent Bernoulli r.v.'s, each with parameter $p$, then $Y = X_1 + \ldots + X_n$ is a binomial r.v. with parameters $(n, p)$. Since $X_i$'s are independent, we can apply the central limit theorem to the r.v. $Z_n$ defined by

$$Z_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{X_i - E(X_i)}{\operatorname{Var}(X_i)}\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{X_i - p}{\sqrt{p(1-p)}}\right) \tag{4.179}$$

Thus, for large $n$, $Z_n$ is normally distributed and

$$P(Z_n \le x) \approx \Phi(x) \tag{4.180}$$

Substituting Eq. (4.179) into Eq. (4.180) gives

$$P\left[\frac{1}{\sqrt{np(1-p)}}\left(\sum_{i=1}^{n}(X_i - p)\right) \le x\right] = P\left[Y \le x\sqrt{np(1-p)} + np\right] \approx \Phi(x)$$

or

$$P(Y \le y) \approx \Phi\left(\frac{y - np}{\sqrt{np(1-p)}}\right)$$

Because we are approximating a discrete distribution by a continuous one, a slightly better approximation is given by

$$P(Y \le y) \approx \Phi\left(\frac{y + \frac{1}{2} - np}{\sqrt{np(1-p)}}\right) \tag{4.181}$$

Formula (4.181) is referred to as a *continuity correction* of Eq. (4.178).

**4.86.** Let $Y$ be a Poisson r.v. with parameter $\lambda$. Using the central limit theorem, derive approximation formula:

$$P(Y \le y) \approx \Phi\left(\frac{y-\lambda}{\sqrt{\lambda}}\right) \tag{4.182}$$

We saw in Prob. 4.71 that if $X_1, \ldots, X_n$ are independent Poisson r.v.'s $X_i$ having parameter $\lambda_i$, then $Y = X_1 + \cdots + X_n$ is also a Poisson r.v. with parameter $\lambda = \lambda_1 + \cdots + \lambda_n$. Using this fact, we can view a Poisson r.v. $Y$ with parameter $\lambda$ as a sum of independent Poisson r.v.'s $X_i$, $i = 1, \cdots, n$, each with parameter $\lambda/n$; that is,

$$Y = X_1 + \cdots + X_n$$
$$E(X_i) = \frac{\lambda}{n} = \mathrm{Var}(X_i)$$

The central limit theorem then implies that the r.v. $Z$ is defined by

$$Z = \frac{Y - E(Y)}{\sqrt{\mathrm{Var}(Y)}} = \frac{Y-\lambda}{\sqrt{\lambda}} \tag{4.183}$$

is approximately normal and

$$P(Z \le z) \approx \Phi(z) \tag{4.184}$$

Substituting Eq. (4.183) into Eq. (4.184) gives

$$P\left(\frac{Y-\lambda}{\sqrt{\lambda}} \le z\right) = P(Y \le \sqrt{\lambda}z + \lambda) \approx \Phi(z)$$

or

$$P(Y \le y) \approx \Phi\left(\frac{y-\lambda}{\sqrt{\lambda}}\right)$$

Again, using a continuity correction, a slightly better approximation is given by

$$P(Y \le y) \approx \Phi\left(\frac{y + \frac{1}{2} - \lambda}{\sqrt{\lambda}}\right) \tag{4.185}$$

## SUPPLEMENTARY PROBLEMS

**4.87.** Let $Y = 2X + 3$. Find the pdf of $Y$ if $X$ is a uniform r.v. over $(-1, 2)$.

**4.88.** Let $X$ be a r.v. with pdf $f_X(x)$. Let $Y = |X|$. Find the pdf of $Y$ in terms of $f_X(x)$.

**4.89.** Let $Y = \sin X$, where $X$ is uniformly distributed over $(0, 2\pi)$. Find the pdf of $Y$.

```
%% Problem 8
close all;
clear all;
clc;
y1 = [];
y2 = [];
y3 = [];
y5 = [];
y10 = [];
y100 = [];

% when N  = 1
%(a)
N_vec= [1 2 3 5 10 100];
N = 10000;
vec = zeros(length(N_vec), N);

for i = 1: length(N_vec)

    for ii = 1: N_vec(i)
        vec(i,:) = vec (i,:) + rand([1,N]);
    end
    ymean(i) = mean(vec(i,:));
    yvar(i) = var(vec(i,:));
end

figure(1);
for a=1:6
    N=[1,2,3,5,10,100];
    val=0:0.01:N(a);
[n xout] = hist(vec(a,:), 10);
area = (xout(2)-xout(1))*sum(n);
subplot(3,2,a);bar(xout, n/area);
hold on;
plot(val, exp((-1*(val-ymean(a)).^2)/2/(yvar(a)))/(sqrt(yvar(a))*sqrt(2*pi)),'--');
hold off;
end

%% Problem 9

N_vec= [1 2 3 5 10 100];
N = 10000;
vec = zeros(length(N_vec), N);

for i = 1: length(N_vec)

    for ii = 1: N_vec(i)
        vec(i,:) = vec (i,:) + random('exponential',1,[1,N]);
    end
    ymean(i) = mean(vec(i,:));
    yvar(i) = var(vec(i,:));
end
figure(2);
for a=1:6
    N=[10,15,15,20,30,150];
```

```
    val=0:0.01:N(a);
[n xout] = hist(vec(a,:), 10);
area = (xout(2)-xout(1))*sum(n);
subplot(3,2,a);bar(xout, n/area);
hold on;
plot(val, exp((-1*(val-ymean(a)).^2)/2/(yvar(a)))/(sqrt(yvar(a))*sqrt(2*pi)),'--');
hold off;
end
```