CHAPTER 2

Solution to Problem 2.1. Let X be the number of points the MIT team earns over the weekend. We have

$$\begin{aligned}
\mathbf{P}(X=0) &= 0.6 \cdot 0.3 = 0.18,\\
\mathbf{P}(X=1) &= 0.4 \cdot 0.5 \cdot 0.3 + 0.6 \cdot 0.5 \cdot 0.7 = 0.27,\\
\mathbf{P}(X=2) &= 0.4 \cdot 0.5 \cdot 0.3 + 0.6 \cdot 0.5 \cdot 0.7 + 0.4 \cdot 0.5 \cdot 0.7 \cdot 0.5 = 0.34,\\
\mathbf{P}(X=3) &= 0.4 \cdot 0.5 \cdot 0.7 \cdot 0.5 + 0.4 \cdot 0.5 \cdot 0.7 \cdot 0.5 = 0.14,\\
\mathbf{P}(X=4) &= 0.4 \cdot 0.5 \cdot 0.7 \cdot 0.5 = 0.07,\\
\mathbf{P}(X>4) &= 0.
\end{aligned}$$

Solution to Problem 2.2. The number of guests that have the same birthday as you is binomial with $p = 1/365$ and $n = 499$. Thus the probability that exactly one other guest has the same birthday is

$$\binom{499}{1} \frac{1}{365} \left(\frac{364}{365}\right)^{498} \approx 0.3486.$$

Let $\lambda = np = 499/365 \approx 1.367$. The Poisson approximation is $e^{-\lambda}\lambda = e^{-1.367} \cdot 1.367 \approx 0.3483$, which closely agrees with the correct probability based on the binomial.

Solution to Problem 2.3. (a) Let L be the duration of the match. If Fischer wins a match consisting of L games, then $L-1$ draws must first occur before he wins. Summing over all possible lengths, we obtain

$$\mathbf{P}(\text{Fischer wins}) = \sum_{l=1}^{10} (0.3)^{l-1}(0.4) = 0.571425.$$

(b) The match has length L with $L < 10$, if and only if $(L-1)$ draws occur, followed by a win by either player. The match has length $L = 10$ if and only if 9 draws occur. The probability of a win by either player is 0.7. Thus

$$p_L(l) = \mathbf{P}(L = l) = \begin{cases} (0.3)^{l-1}(0.7), & l = 1, \ldots, 9,\\ (0.3)^9, & l = 10,\\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 2.4. (a) Let X be the number of modems in use. For $k < 50$, the probability that $X = k$ is the same as the probability that k out of 1000 customers need a connection:

$$p_X(k) = \binom{1000}{k} (0.01)^k (0.99)^{1000-k}, \qquad k = 0, 1, \ldots, 49.$$

The probability that $X = 50$, is the same as the probability that 50 or more out of 1000 customers need a connection:

$$p_X(50) = \sum_{k=50}^{1000} \binom{1000}{k} (0.01)^k (0.99)^{1000-k}.$$

(b) By approximating the binomial with a Poisson with parameter $\lambda = 1000 \cdot 0.01 = 10$, we have

$$p_X(k) = e^{-10} \frac{10^k}{k!}, \qquad k = 0, 1, \ldots, 49,$$

$$p_X(50) = \sum_{k=50}^{1000} e^{-10} \frac{10^k}{k!}.$$

(c) Let A be the event that there are more customers needing a connection than there are modems. Then,

$$\mathbf{P}(A) = \sum_{k=51}^{1000} \binom{1000}{k} (0.01)^k (0.99)^{1000-k}.$$

With the Poisson approximation, $\mathbf{P}(A)$ is estimated by

$$\sum_{k=51}^{1000} e^{-10} \frac{10^k}{k!}.$$

Solution to Problem 2.5. (a) Let X be the number of packets stored at the end of the first slot. For $k < b$, the probability that $X = k$ is the same as the probability that k packets are generated by the source:

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \qquad k = 0, 1, \ldots, b-1,$$

while

$$p_X(b) = \sum_{k=b}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = 1 - \sum_{k=0}^{b-1} e^{-\lambda} \frac{\lambda^k}{k!}.$$

Let Y be the number of number of packets stored at the end of the second slot. Since $\min\{X, c\}$ is the number of packets transmitted in the second slot, we have $Y = X - \min\{X, c\}$. Thus,

$$p_Y(0) = \sum_{k=0}^{c} p_X(k) = \sum_{k=0}^{c} e^{-\lambda} \frac{\lambda^k}{k!},$$

$$p_Y(k) = p_X(k+c) = e^{-\lambda} \frac{\lambda^{k+c}}{(k+c)!}, \qquad k = 1, \ldots, b-c-1,$$

$$p_Y(b-c) = p_X(b) = 1 - \sum_{k=0}^{b-1} e^{-\lambda} \frac{\lambda^k}{k!}.$$

(b) The probability that some packets get discarded during the first slot is the same as the probability that more than b packets are generated by the source, so it is equal to

$$\sum_{k=b+1}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!},$$

or

$$1 - \sum_{k=0}^{b} e^{-\lambda} \frac{\lambda^k}{k!}.$$

Solution to Problem 2.6. We consider the general case of part (b), and we show that $p > 1/2$ is a necessary and sufficient condition for $n = 2k+1$ games to be better than $n = 2k-1$ games. To prove this, let N be the number of Celtics' wins in the first $2k-1$ games. If A denotes the event that the Celtics win with $n = 2k+1$, and B denotes the event that the Celtics win with $n = 2k-1$, then

$$\mathbf{P}(A) = \mathbf{P}(N \geq k+1) + \mathbf{P}(N = k) \cdot \big(1 - (1-p)^2\big) + \mathbf{P}(N = k-1) \cdot p^2,$$

$$\mathbf{P}(B) = \mathbf{P}(N \geq k) = \mathbf{P}(N = k) + \mathbf{P}(N \geq k+1),$$

and therefore

$$\begin{aligned} \mathbf{P}(A) - \mathbf{P}(B) &= \mathbf{P}(N = k-1) \cdot p^2 - \mathbf{P}(N = k) \cdot (1-p)^2 \\ &= \binom{2k-1}{k-1} p^{k-1} (1-p)^k p^2 - \binom{2k-1}{k} (1-p)^2 p^k (1-p)^{k-1} \\ &= \frac{(2k-1)!}{(k-1)!\,k!} p^k (1-p)^k (2p-1). \end{aligned}$$

It follows that $\mathbf{P}(A) > \mathbf{P}(B)$ if and only if $p > \frac{1}{2}$. Thus, a longer series is better for the better team.

Solution to Problem 2.7. Let random variable X be the number of trials you need to open the door, and let K_i be the event that the ith key selected opens the door.

(a) In case (1), we have

$$\begin{aligned} p_X(1) &= \mathbf{P}(K_1) = \frac{1}{5}, \\ p_X(2) &= \mathbf{P}(K_1^c)\mathbf{P}(K_2 \,|\, K_1^c) = \frac{4}{5} \cdot \frac{1}{4} = \frac{1}{5}, \\ p_X(3) &= \mathbf{P}(K_1^c)\mathbf{P}(K_2^c \,|\, K_1^c)\mathbf{P}(K_3 \,|\, K_1^c \cap K_2^c) = \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{5}. \end{aligned}$$

Proceeding similarly, we see that the PMF of X is

$$p_X(x) = \frac{1}{5}, \qquad x = 1, 2, 3, 4, 5.$$

If $k \le k^*$, then $k \le (n+1)p$, or equivalently $k - kp \le (n+1)p - kp$, so that the above ratio is greater than or equal to 1. It follows that $p_X(k)$ is monotonically nondecreasing. If $k > k^*$, the ratio is less than one, and $p_X(k)$ is monotonically decreasing, as required.

Solution to Problem 2.10. Using the expression for the Poisson PMF, we have, for $k \ge 1$,

$$\frac{p_X(k)}{p_X(k-1)} = \frac{\lambda^k \cdot e^{-\lambda}}{k!} \cdot \frac{(k-1)!}{\lambda^{k-1} \cdot e^{-\lambda}} = \frac{\lambda}{k}.$$

Thus if $k \le \lambda$ the ratio is greater or equal to 1, and it follows that $p_X(k)$ is monotonically increasing. Otherwise, the ratio is less than one, and $p_X(k)$ is monotonically decreasing, as required.

Solution to Problem 2.13. We will use the PMF for the number of girls among the natural children together with the formula for the PMF of a function of a random variable. Let N be the number of natural children that are girls. Then N has a binomial PMF

$$p_N(k) = \begin{cases} \binom{5}{k} \cdot \left(\frac{1}{2}\right)^5, & \text{if } 0 \le k \le 5, \\ 0, & \text{otherwise.} \end{cases}$$

Let G be the number of girls out of the 7 children, so that $G = N + 2$. By applying the formula for the PMF of a function of a random variable, we have

$$p_G(g) = \sum_{\{n \mid n+2=g\}} p_N(n) = p_N(g-2).$$

Thus

$$p_G(g) = \begin{cases} \binom{5}{g-2} \cdot \left(\frac{1}{2}\right)^5, & \text{if } 2 \le g \le 7, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 2.14. (a) Using the formula $p_Y(y) = \sum_{\{x \mid x \bmod(3)=y\}} p_X(x)$, we obtain

$$\begin{aligned} p_Y(0) &= p_X(0) + p_X(3) + p_X(6) + p_X(9) = 4/10, \\ p_Y(1) &= p_X(1) + p_X(4) + p_X(7) = 3/10, \\ p_Y(2) &= p_X(2) + p_X(5) + p_X(8) = 3/10, \\ p_Y(y) &= 0, \qquad \text{if } y \notin \{0,1,2\}. \end{aligned}$$

(b) Similarly, using the formula $p_Y(y) = \sum_{\{x \mid 5 \bmod(x+1)=y\}} p_X(x)$, we obtain

$$p_Y(y) = \begin{cases} 2/10, & \text{if } y = 0, \\ 2/10, & \text{if } y = 1, \\ 1/10, & \text{if } y = 2, \\ 5/10, & \text{if } y = 5, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 2.15. The random variable Y takes the values $k \ln a$, where $k = 1, \ldots, n$, if and only if $X = a^k$ or $X = a^{-k}$. Furthermore, Y takes the value 0, if and only if $X = 1$. Thus, we have

$$p_Y(y) = \begin{cases} \dfrac{2}{2n+1}, & \text{if } y = \ln a,\ 2\ln a, \ldots, k \ln a, \\ \dfrac{1}{2n+1}, & \text{if } y = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 2.16. (a) The scalar a must satisfy

$$1 = \sum_x p_X(x) = \frac{1}{a} \sum_{x=-3}^{3} x^2,$$

so

$$a = \sum_{x=-3}^{3} x^2 = (-3)^2 + (-2)^2 + (-1)^2 + 1^2 + 2^2 + 3^2 = 28.$$

We also have $\mathbf{E}[X] = 0$ because the PMF is symmetric around 0.

(b) If $z \in \{1, 4, 9\}$, then

$$p_Z(z) = p_X(\sqrt{z}) + p_X(-\sqrt{z}) = \frac{z}{28} + \frac{z}{28} = \frac{z}{14}.$$

Otherwise $p_Z(z) = 0$.

(c) $\mathrm{var}(X) = \mathbf{E}[Z] = \sum_z z p_Z(z) = \sum_{z \in \{1,4,9\}} \frac{z^2}{14} = 7.$

(d) We have

$$\begin{aligned} \mathrm{var}(X) &= \sum_x (x - \mathbf{E}[X])^2 p_X(x) \\ &= 1^2 \cdot \big(p_X(-1) + p_X(1)\big) + 2^2 \cdot \big(p_X(-2) + p_X(2)\big) + 3^2 \cdot \big(p_X(-3) + p_X(3)\big) \\ &= 2 \cdot \frac{1}{28} + 8 \cdot \frac{4}{28} + 18 \cdot \frac{9}{28} \\ &= 7. \end{aligned}$$

Solution to Problem 2.17. If X is the temperature in Celsius, the temperature in Fahrenheit is $Y = 32 + 9X/5$. Therefore,

$$\mathbf{E}[Y] = 32 + 9\mathbf{E}[X]/5 = 32 + 18 = 50.$$

Also

$$\mathrm{var}(Y) = (9/5)^2 \mathrm{var}(X),$$

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trials = 100000;

%Binomial
figure (1);
X_vector = random('Binomial', 20, 0.2, [1, trials]);
for i =1:21
    count (i, 1) = i-1 ;
    count (i, 2) = sum( (i-1) == X_vector );
end

pvec = count(:, 2)/trials;
bar ( count(:, 1), pvec);

clear count;

%Geometric
figure (2);
Y_vector = random('Geometric', 0.1, [1,trials]);
for i =1:21
    count (i, 1) = i-1 ;
    count (i, 2) = sum( (i-1) == Y_vector );
end
pvec = count(:, 2)/trials;
bar ( count(:, 1), pvec);
clear count;

%Poisson
figure (3);
Z_vector = random('Poisson', 3, [1,trials]);
for i =1:21
    count (i, 1) = i-1 ;
    count (i, 2) = sum( (i-1) == Z_vector );
end
pvec = count(:, 2)/trials;
bar ( count(:, 1), pvec);
clear count;

%Last Question

figure (4);
X1 = random('binomial', 3, 0.25, [1,trials]);
X2 = random('binomial', 4, 0.5, [1,trials]);
Y1 = (X1 + X2 );
for i =1:11
    count (i, 1) = i-1 ;
    count (i, 2) = sum( (i-1) == Y1 );
end
pvec = count(:, 2)/trials;
bar ( count(:, 1), pvec)
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X_1 — binomial RV, Parameters $n=3$ $p=1/4$

$$P_{X_1}(k) = \binom{3}{k} (1/4)^k (1-1/4)^{3-k} \quad k=0,1,2,3$$

$P_{X_1}(0) = 0.4219$ $\quad P_{X_1}(2) = 0.1406$

$P_{X_1}(1) = 0.4219$ $\quad P_{X_1}(3) = 0.0156$

X_2 - binomial RV with parameters $n=4$, $p=1/2$

$$P_{X_2}(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$P_{X_2}(0) = 0.0625$ $\quad P_{X_2}(3) = 0.25$

$P_{X_2}(1) = 0.25$ $\quad P_{X_2}(4) = 0.0625$

$P_{X_2}(2) = 0.375$

$$P_Y(0) = P(X_1=0, X_2=0) = P(\{X_1=0\} \cap \{X_2=0\})$$
$$= P_{X_1}(0)\, P_{X_2}(0) = 0.0264$$

Now,

marginal pmf $P_{X_2}(x_2)$	0	1	2	3
4	0.0264	0.0264	0.0088	0.0010
3	0.1055	0.1055	0.0352	0.0039
2	0.1582	0.1582	0.0527	0.0058
1	0.1055	0.1055	0.0352	0.0039
0	0.0264	0.0264	0.0088	0.0010

$P_{X_1}(x_1)$

$P_Y(y) =$

0.0264	$y=0$
0.1319	$y=1$
0.2725	$y=2$
0.2999	$y=3$
0.1885	$y=4$
0.0674	$y=5$
0.0127	$y=6$
0.0010	$y=7$