Solution to Problem 2.15. The random variable Y takes the values $k \ln a$, where $k = 1, \ldots, n$, if and only if $X = a^k$ or $X = a^{-k}$. Furthermore, Y takes the value 0, if and only if $X = 1$. Thus, we have

$$p_Y(y) = \begin{cases} \dfrac{2}{2n+1}, & \text{if } y = \ln a,\ 2\ln a, \ldots, k \ln a, \\ \dfrac{1}{2n+1}, & \text{if } y = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 2.16. (a) The scalar a must satisfy

$$1 = \sum_x p_X(x) = \frac{1}{a} \sum_{x=-3}^{3} x^2,$$

so

$$a = \sum_{x=-3}^{3} x^2 = (-3)^2 + (-2)^2 + (-1)^2 + 1^2 + 2^2 + 3^2 = 28.$$

We also have $\mathbf{E}[X] = 0$ because the PMF is symmetric around 0.

(b) If $z \in \{1, 4, 9\}$, then

$$p_Z(z) = p_X(\sqrt{z}) + p_X(-\sqrt{z}) = \frac{z}{28} + \frac{z}{28} = \frac{z}{14}.$$

Otherwise $p_Z(z) = 0$.

(c) $\text{var}(X) = \mathbf{E}[Z] = \sum_z z p_Z(z) = \sum_{z \in \{1,4,9\}} \frac{z^2}{14} = 7.$

(d) We have

$$\begin{aligned} \text{var}(X) &= \sum_x (x - \mathbf{E}[X])^2 p_X(x) \\ &= 1^2 \cdot \big(p_X(-1) + p_X(1)\big) + 2^2 \cdot \big(p_X(-2) + p_X(2)\big) + 3^2 \cdot \big(p_X(-3) + p_X(3)\big) \\ &= 2 \cdot \frac{1}{28} + 8 \cdot \frac{4}{28} + 18 \cdot \frac{9}{28} \\ &= 7. \end{aligned}$$

Solution to Problem 2.17. If X is the temperature in Celsius, the temperature in Fahrenheit is $Y = 32 + 9X/5$. Therefore,

$$\mathbf{E}[Y] = 32 + 9\mathbf{E}[X]/5 = 32 + 18 = 50.$$

Also

$$\text{var}(Y) = (9/5)^2 \text{var}(X),$$

where var(X), the square of the given standard deviation of X, is equal to 100. Thus, the standard deviation of Y is $(9/5) \cdot 10 = 18$. Hence a normal day in Fahrenheit is one for which the temperature is in the range $[32, 68]$.

Solution to Problem 2.18. We have

$$p_X(x) = \begin{cases} 1/(b-a+1), & \text{if } x = 2^k, \text{ where } a \le k \le b, \ k \text{ integer}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\mathbf{E}[X] = \sum_{k=a}^{b} \frac{1}{b-a+1} 2^k = \frac{2^a}{b-a+1}(1 + 2 + \cdots + 2^{b-a}) = \frac{2^{b+1} - 2^a}{b-a+1}.$$

Similarly,

$$\mathbf{E}[X^2] = \sum_{k=a}^{b} \frac{1}{b-a+1} (2^k)^2 = \frac{4^{b+1} - 4^a}{3(b-a+1)},$$

and finally

$$\text{var}(X) = \frac{4^{b+1} - 4^a}{3(b-a+1)} - \left(\frac{2^{b+1} - 2^a}{b-a+1}\right)^2.$$

Solution to Problem 2.19. We will find the expected gain for each strategy, by computing the expected number of questions until we find the prize.

(a) With this strategy, the probability of finding the location of the prize with i questions, where $i = 1, \ldots, 8$, is $1/10$. The probability of finding the location with 9 questions is $2/10$. Therefore, the expected number of questions is

$$\frac{2}{10} \cdot 9 + \frac{1}{10} \sum_{i=1}^{8} i = 5.4.$$

(b) It can be checked that for 4 of the 10 possible box numbers, exactly 4 questions will be needed, whereas for 6 of the 10 numbers, 3 questions will be needed. Therefore, with this strategy, the expected number of questions is

$$\frac{4}{10} \cdot 4 + \frac{6}{10} \cdot 3 = 3.4.$$

Solution to Problem 2.20. The number C of candy bars you need to eat is a geometric random variable with parameter p. Thus the mean is $\mathbf{E}[C] = 1/p$, and the variance is $\text{var}(C) = (1-p)/p^2$.

Solution to Problem 2.21. The expected value of the gain for a single game is infinite since if X is your gain, then

$$\mathbf{E}[X] = \sum_{k=1}^{\infty} 2^k \cdot 2^{-k} = \sum_{k=1}^{\infty} 1 = \infty.$$

segment. For the case where $k = 2$, there is only one (hence $k-1$) possible sequence that leads to the event $\{X = k\}$, namely the sequence HT. Therefore, for any $k \geq 2$,

$$\mathbf{P}(X = k) = (k-1)(1/2)^k.$$

It follows that

$$p_X(k) = \begin{cases} (k-1)(1/2)^k, & \text{if } k \geq 2, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathbf{E}[X] = \sum_{k=2}^{\infty} k(k-1)(1/2)^k = \sum_{k=1}^{\infty} k(k-1)(1/2)^k = \sum_{k=1}^{\infty} k^2(1/2)^k - \sum_{k=1}^{\infty} k(1/2)^k = 6-2 = 4.$$

We have used here the equalities

$$\sum_{k=1}^{\infty} k(1/2)^k = \mathbf{E}[Y] = 2,$$

and

$$\sum_{k=1}^{\infty} k^2(1/2)^k = \mathbf{E}[Y^2] = \text{var}(Y) + \big(\mathbf{E}[Y]\big)^2 = 2 + 2^2 = 6,$$

where Y is a geometric random variable with parameter $p = 1/2$.

Solution to Problem 2.24. (a) There are 21 integer pairs (x, y) in the region

$$R = \Big\{(x, y) \mid -2 \leq x \leq 4, \ -1 \leq y - x \leq 1\Big\},$$

so that the joint PMF of X and Y is

$$p_{X,Y}(x, y) = \begin{cases} 1/21, & \text{if } (x, y) \text{ is in } R, \\ 0, & \text{otherwise.} \end{cases}$$

For each x in the range $[-2, 4]$, there are three possible values of Y. Thus, we have

$$p_X(x) = \begin{cases} 3/21, & \text{if } x = -2, -1, 0, 1, 2, 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$

The mean of X is the midpoint of the range $[-2, 4]$:

$$\mathbf{E}[X] = 1.$$

The marginal PMF of Y is obtained by using the tabular method. We have

$$p_Y(y) = \begin{cases} 1/21, & \text{if } y = -3, \\ 2/21, & \text{if } y = -2, \\ 3/21, & \text{if } y = -1, 0, 1, 2, 3, \\ 2/21, & \text{if } y = 4, \\ 1/21, & \text{if } y = 5, \\ 0, & \text{otherwise.} \end{cases}$$

The mean of Y is

$$\mathbf{E}[Y] = \frac{1}{21} \cdot (-3+5) + \frac{2}{21} \cdot (-2+4) + \frac{3}{21} \cdot (-1+1+2+3) = 1.$$

(b) The profit is given by

$$P = 100X + 200Y,$$

so that

$$\mathbf{E}[P] = 100 \cdot \mathbf{E}[X] + 200 \cdot \mathbf{E}[Y] = 100 \cdot 1 + 200 \cdot 1 = 300.$$

Solution to Problem 2.25. (a) Since all possible values of (I, J) are equally likely, we have

$$p_{I,J}(i,j) = \begin{cases} \frac{1}{\sum_{k=1}^{n} m_k}, & \text{if } j \leq m_i, \\ 0, & \text{otherwise.} \end{cases}$$

The marginal PMFs are given by

$$p_I(i) = \sum_{j=1}^{m} p_{I,J}(i,j) = \frac{m_i}{\sum_{k=1}^{n} m_k}, \qquad i = 1, \ldots, n,$$

$$p_J(j) = \sum_{i=1}^{n} p_{I,J}(i,j) = \frac{l_j}{\sum_{k=1}^{n} m_k}, \qquad j = 1, \ldots, m,$$

where l_j is the number of students that have answered question j, i.e., students i with $j \leq m_i$.

(b) The expected value of the score of student i is the sum of the expected values $p_{ij}a + (1-p_{ij})b$ of the scores on questions j with $j = 1, \ldots, m_i$, i.e.,

$$\sum_{j=1}^{m_i} \big(p_{ij}a + (1-p_{ij})b\big).$$

Solution to Problem 2.26. (a) The possible values of the random variable X are the ten numbers $101, \ldots, 110$, and the PMF is given by

$$p_X(k) = \begin{cases} \mathbf{P}(X > k-1) - \mathbf{P}(X > k), & \text{if } k = 101, \ldots 110, \\ 0, & \text{otherwise.} \end{cases}$$

We have $\mathbf{P}(X > 100) = 1$ and for $k = 101, \ldots 110$,

$$\begin{aligned} \mathbf{P}(X > k) &= \mathbf{P}(X_1 > k, X_2 > k, X_3 > k) \\ &= \mathbf{P}(X_1 > k)\,\mathbf{P}(X_2 > k)\,\mathbf{P}(X_3 > k) \\ &= \frac{(110-k)^3}{10^3}. \end{aligned}$$

It follows that

$$p_X(k) = \begin{cases} \dfrac{(111-k)^3 - (110-k)^3}{10^3}, & \text{if } k = 101, \dots 110, \\ 0, & \text{otherwise.} \end{cases}$$

(An alternative solution is based on the notion of a CDF, which will be introduced in Chapter 3.)

(b) Since X_i is uniformly distributed over the integers in the range $[101, 110]$, we have $\mathbf{E}[X_i] = (101 + 110)/2 = 105.5$. The expected value of X is

$$\mathbf{E}[X] = \sum_{k=-\infty}^{\infty} k \cdot p_X(k) = \sum_{k=101}^{110} k \cdot p_x(k) = \sum_{k=101}^{110} k \cdot \frac{(111-k)^3 - (110-k)^3}{10^3}.$$

The above expression can be evaluated to be equal to 103.025. The expected improvement is therefore 105.5 - 103.025 = 2.475.

Solution to Problem 2.31. The marginal PMF p_Y is given by the binomial formula

$$p_Y(y) = \binom{4}{y} \left(\frac{1}{6}\right)^y \left(\frac{5}{6}\right)^{4-y}, \qquad y = 0, 1, \dots, 4.$$

To compute the conditional PMF $p_{X|Y}$, note that given that $Y = y$, X is the number of 1's in the remaining $4 - y$ rolls, each of which can take the 5 values $1, 3, 4, 5, 6$ with equal probability $1/5$. Thus, the conditional PMF $p_{X|Y}$ is binomial with parameters $4 - y$ and $p = 1/5$:

$$p_{X|Y}(x \,|\, y) = \binom{4-y}{x} \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{4-y-x},$$

for all nonnegative integers x and y such that $0 \le x + y \le 4$. The joint PMF is now given by

$$\begin{aligned} p_{X,Y}(x, y) &= p_Y(y) p_{X|Y}(x \,|\, y) \\ &= \binom{4}{y} \left(\frac{1}{6}\right)^y \left(\frac{5}{6}\right)^{4-y} \binom{4-y}{x} \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{4-y-x}, \end{aligned}$$

for all nonnegative integers x and y such that $0 \le x + y \le 4$. For other values of x and y, we have $p_{X,Y}(x, y) = 0$.

Solution to Problem 2.32. Let X_i be the random variable taking the value 1 or 0 depending on whether the first partner of the ith couple has survived or not. Let Y_i be the corresponding random variable for the second partner of the ith couple. Then, we have $S = \sum_{i=1}^{m} X_i Y_i$, and by using the total expectation theorem,

$$\begin{aligned} \mathbf{E}[S \,|\, A = a] &= \sum_{i=1}^{m} \mathbf{E}[X_i Y_i \,|\, A = a] \\ &= m \mathbf{E}[X_1 Y_1 \,|\, A = a] \\ &= m \mathbf{E}[Y_1 = 1 \,|\, X_1 = 1,\, A = a] \mathbf{P}(X_1 = 1 \,|\, A = a) \\ &= m \mathbf{P}(Y_1 = 1 \,|\, X_1 = 1,\, A = a) \mathbf{P}(X_1 = 1 \,|\, A = a). \end{aligned}$$

...ional Distributions

. Let X be a Poisson r.v. with parameter λ. Find the conditional pmf of X given $B = (X$ is eve

From Eq. (2.48), the pdf of X is

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!} \qquad k = 0, 1, \ldots$$

Then the probability of event B is

$$P(B) = P(X = 0, 2, 4, \ldots) = \sum_{k=\text{even}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}$$

Let $A = \{X \text{ is odd}\}$. Then the probability of event A is

$$P(A) = P(X = 1, 3, 5, \ldots) = \sum_{k=\text{odd}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}$$

Now

$$\sum_{k=\text{even}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} + \sum_{k=\text{odd}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = 1$$

$$\sum_{k=\text{even}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} - \sum_{k=\text{odd}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!} = e^{-\lambda}e^{-\lambda} = e^{-2\lambda}$$

Hence, adding Eqs. (2.120) and (2.121), we obtain

$$P(B) = \sum_{k=\text{even}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = \frac{1}{2}(1 + e^{-2\lambda})$$

Now, by Eq. (2.81), the pmf of X given B is

$$p_X(k|B) = \frac{P\{(X = k) \cap B\}}{P(B)}$$

If k is even, $(X = k) \subset B$ and $(X = k) \cap B = (X = k)$. If k is odd, $(X = k) \cap B = \varnothing$. Hence,

$$p_X(k \mid B) = \begin{cases} \dfrac{P(X = k)}{P(B)} = \dfrac{2e^{-\lambda}\lambda^k}{(1 + e^{-2\lambda})k!} = & k \text{ even} \\[2ex] \dfrac{P(\varnothing)}{P(B)} = 0 & k \text{ odd} \end{cases}$$

2.54. Show that the conditional cdf and pdf of X given the event $B = (a < X \le b)$ are as follows:

$$F_X(x \mid a < X \le b) = \begin{cases} 0 & x \le a \\ \dfrac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} & a < x \le b \\ 1 & x > b \end{cases}$$

$$f_X(x \mid a < X \le b) = \begin{cases} 0 & x \le a \\ \dfrac{f_X(x)}{\int_a^b f_X(\xi)\, d\xi} & a < x \le b \\ 0 & x > b \end{cases}$$

```matlab
%%
clear all;
close all;
%%
load burgerfry.mat

%%BF=zeros(6,4);
%Y-value: Fries 4
BF = [ 0 0 0 0;... %
       0 0 0 0;... %
       0 0 0 0;... %
       0 0 0 0;... % X-Values:burger from 1 to 6
       0 0 0 0;... %
       0 0 0 0;... %
       ];
%%
%Part 1
%%
Probability = 1/10000;
for i = 1:10000;
   row = outcomes(i,1); % burger
   col = outcomes(i,2); % fry

   BF(row, col) = BF(row,col) + Probability;

end
figure(1);
bar3(BF);
xlabel('# of FRY');
ylabel('# of BURGER');
zlabel('Probability Mass');
title('Original Joint PMF');
%%
%Part 2: three burgers and two servings of fries
%%
answer = BF(3,2)
%%
%Part 3:marginal PMF for the number of burgers
%%
figure(2);
marg_fries = sum(BF)
marg_burgers = sum(BF')

subplot(2,1,1);
bar(marg_burgers);
title('Marginal PMF for X');
xlabel('# of fries');
ylabel('Probability Mass');
%%
% Part 4: marginal PMF for the number of fries servings
%%
subplot(2,1,2);
bar(marg_fries);
title('Marginal PMF for Y');
xlabel('# of burgers');
```

```
ylabel('Probability Mass');
%%
%Part 5:expected number of burgers
%%
Burger=[1:6];
expectation_burger=sum(marg_burgers.*Burger);


%%
%Part 6: expected number of fries
%%
fries=[1:4];
expectation_fries=sum(marg_fries.*fries);

%%
%Part 7: burgers cost $2.00 and fries servings cost $1.00, what is the
%expected amount of money that you will obtain
%%
total_money=2*expectation_burger + 1*expectation_fries;

%%
% Part 8:customer buys two fries servings, what is the PMF of the number of
% burgers that he will buy?
%%
cond=BF(:,2)/sum(BF(:,2));
figure(3);
bar3(cond);
ylabel('# of burgur');
zlabel('Probability Mass');
title('at 2 fries');
```