We have

$$\mathbf{P}(Y_1 = 1 \mid X_1 = 1,\, A = a) = \frac{a-1}{2m-1}, \qquad \mathbf{P}(X_1 = 1 \mid A = a) = \frac{a}{2m}.$$

Thus

$$\mathbf{E}[S \mid A = a] = m\, \frac{a-1}{2m-1} \cdot \frac{a}{2m} = \frac{a(a-1)}{2(2m-1)}.$$

Note that $\mathbf{E}[S \mid A = a]$ does not depend on p.

Solution to Problem 2.38. (a) Let X be the number of red lights that Alice encounters. The PMF of X is binomial with $n = 4$ and $p = 1/2$. The mean and the variance of X are $\mathbf{E}[X] = np = 2$ and $\text{var}(X) = np(1-p) = 4 \cdot (1/2) \cdot (1/2) = 1$.

(b) The variance of Alice's commuting time is the same as the variance of the time by which Alice is delayed by the red lights. This is equal to the variance of $2X$, which is $4\text{var}(X) = 4$.

Solution to Problem 2.39. Let X_i be the number of eggs Harry eats on day i. Then, the X_i are independent random variables, uniformly distributed over the set $\{1, \ldots, 6\}$. We have $X = \sum_{i=1}^{10} X_i$, and

$$\mathbf{E}[X] = \mathbf{E}\left(\sum_{i=1}^{10} X_i\right) = \sum_{i=1}^{10} \mathbf{E}[X_i] = 35.$$

Similarly, we have

$$\text{var}(X) = \text{var}\left(\sum_{i=1}^{10} X_i\right) = \sum_{i=1}^{10} \text{var}(X_i),$$

since the X_i are independent. Using the formula of Example 2.6, we have

$$\text{var}(X_i) = \frac{(6-1)(6-1+2)}{12} \approx 2.9167,$$

so that $\text{var}(X) \approx 29.167$.

Solution to Problem 2.40. Associate a success with a paper that receives a grade that has not been received before. Let X_i be the number of papers between the ith success and the $(i+1)$st success. Then we have $X = 1 + \sum_{i=1}^{5} X_i$ and hence

$$\mathbf{E}[X] = 1 + \sum_{i=1}^{5} \mathbf{E}[X_i].$$

After receiving $i-1$ different grades so far ($i-1$ successes), each subsequent paper has probability $(6-i)/6$ of receiving a grade that has not been received before. Therefore, the random variable X_i is geometric with parameter $p_i = (6-i)/6$, so $\mathbf{E}[X_i] = 6/(6-i)$. It follows that

$$\mathbf{E}[X] = 1 + \sum_{i=1}^{5} \frac{6}{6-i} = 1 + 6 \sum_{i=1}^{5} \frac{1}{i} = 14.7.$$

Solution to Problem 2.41. (a) The PMF of X is the binomial PMF with parameters $p = 0.02$ and $n = 250$. The mean is $\mathbf{E}[X] = np = 250 \cdot 0.02 = 5$. The desired probability is

$$\mathbf{P}(X = 5) = \binom{250}{5} (0.02)^5 (0.98)^{245} = 0.1773.$$

(b) The Poisson approximation has parameter $\lambda = np = 5$, so the probability in (a) is approximated by

$$e^{-\lambda} \frac{\lambda^5}{5!} = 0.1755.$$

(c) Let Y be the amount of money you pay in traffic tickets during the year. Then

$$\mathbf{E}[Y] = \sum_{i=1}^{5} 50 \cdot \mathbf{E}[Y_i],$$

where Y_i is the amount of money you pay on the ith day. The PMF of Y_i is

$$\mathbf{P}(Y_i = y) = \begin{cases} 0.98, & \text{if } y = 0, \\ 0.01, & \text{if } y = 10, \\ 0.006, & \text{if } y = 20, \\ 0.004, & \text{if } y = 50. \end{cases}$$

The mean is

$$\mathbf{E}[Y_i] = 0.01 \cdot 10 + 0.006 \cdot 20 + 0.004 \cdot 50 = 0.42.$$

The variance is

$$\text{var}(Y_i) = \mathbf{E}[Y_i^2] - \big(\mathbf{E}[Y_i]\big)^2 = 0.01 \cdot (10)^2 + 0.006 \cdot (20)^2 + 0.004 \cdot (50)^2 - (0.42)^2 = 13.22.$$

The mean of Y is

$$\mathbf{E}[Y] = 250 \cdot \mathbf{E}[Y_i] = 105,$$

and using the independence of the random variables Y_i, the variance of Y is

$$\text{var}(Y) = 250 \cdot \text{var}(Y_i) = 3,305.$$

(d) The variance of the sample mean is

$$\frac{p(1-p)}{250}$$

so assuming that $|p - \hat{p}|$ is within 5 times the standard deviation, the possible values of p are those that satisfy $p \in [0, 1]$ and

$$(p - 0.02)^2 \leq \frac{25p(1-p)}{250}.$$

This is a quadratic inequality that can be solved for the interval of values of p. After some calculation, the inequality can be written as $275p^2 - 35p + 0.1 \leq 0$, which holds if and only if $p \in [0.0025, 0.1245]$.

Solution to Problem 2.42. (a) Noting that

$$\mathbf{P}(X_i = 1) = \frac{\text{Area}(S)}{\text{Area}\big([0,1] \times [0,1]\big)} = \text{Area}(S),$$

we obtain

$$\mathbf{E}[S_n] = \mathbf{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}[X_i] = \mathbf{E}[X_i] = \text{Area}(S),$$

and

$$\text{var}(S_n) = \text{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} \text{var}(X_i) = \frac{1}{n}\text{var}(X_i) = \frac{1}{n}\big(1 - \text{Area}(S)\big)\text{Area}(S),$$

which tends to zero as n tends to infinity.

(b) We have

$$S_n = \frac{n-1}{n} S_{n-1} + \frac{1}{n} X_n.$$

(c) We can generate S_{10000} (up to a certain precision) as follows :

1. Initialize S to zero.
2. For $i = 1$ to 10000
3. Randomly select two real numbers a and b (up to a certain precision) independently and uniformly from the interval $[0, 1]$.
4. If $(a - 0.5)^2 + (b - 0.5)^2 < 0.25$, set x to 1 else set x to 0.
5. Set $S := (i - 1)S/i + x/i$.
6. Return S.

By running the above algorithm, a value of S_{10000} equal to 0.7783 was obtained (the exact number depends on the random number generator). We know from part (a) that the variance of S_n tends to zero as n tends to infinity, so the obtained value of S_{10000} is an approximation of $\mathbf{E}[S_{10000}]$. But $\mathbf{E}[S_{10000}] = \text{Area}(S) = \pi/4$, this leads us to the following approximation of π:

$$4 \cdot 0.7783 = 3.1132.$$

(d) We only need to modify the test done at step 4. We have to test whether or not $0 \leq \cos \pi a + \sin \pi b \leq 1$. The obtained approximation of the area was 0.3755.

CHAPTER 3

Solution to Problem 3.1. The random variable $Y = g(X)$ is discrete and its PMF is given by

$$p_Y(1) = \mathbf{P}(X \le 1/3) = 1/3, \qquad p_Y(2) = 1 - p_Y(1) = 2/3.$$

Thus,

$$\mathbf{E}[Y] = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 2 = \frac{5}{3}.$$

The same result is obtained using the expected value rule:

$$\mathbf{E}[Y] = \int_0^1 g(x) f_X(x)\, dx = \int_0^{1/3} dx + \int_{1/3}^1 2\, dx = \frac{5}{3}.$$

Solution to Problem 3.2. We have

$$\int_{-\infty}^{\infty} f_X(x) dx = \int_{-\infty}^{\infty} \frac{\lambda}{2} e^{-\lambda|x|}\, dx = 2 \cdot \frac{1}{2} \int_0^{\infty} \lambda e^{-\lambda x}\, dx = 2 \cdot \frac{1}{2} = 1,$$

where we have used the fact $\int_0^\infty \lambda e^{-\lambda x} dx = 1$, i.e., the normalization property of the exponential PDF. By symmetry of the PDF, we have $\mathbf{E}[X] = 0$. We also have

$$\mathbf{E}[X^2] = \int_{-\infty}^{\infty} x^2 \frac{\lambda}{2} e^{-\lambda|x|} dx = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx = \frac{2}{\lambda^2},$$

where we have used the fact that the second moment of the exponential PDF is $2/\lambda^2$. Thus

$$\text{var}(X) = \mathbf{E}[X^2] - \big(\mathbf{E}[X]\big)^2 = 2/\lambda^2.$$

Solution to Problem 3.5. Let $A = bh/2$ be the area of the given triangle, where b is the length of the base, and h is the height of the triangle. From the randomly chosen point, draw a line parallel to the base, and let A_x be the area of the triangle thus formed. The height of this triangle is $h - x$ and its base has length $b(h - x)/h$. Thus $A_x = b(h - x)^2/(2h)$. For $x \in [0, h]$, we have

$$F_X(x) = 1 - \mathbf{P}(X > x) = 1 - \frac{A_x}{A} = 1 - \frac{b(h - x)^2/(2h)}{bh/2} = 1 - \left(\frac{h - x}{h}\right)^2,$$

while $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x > h$.

The PDF is obtained by differentiating the CDF. We have

$$f_X(x) = \frac{dF_X}{dx}(x) = \begin{cases} \dfrac{2(h - x)}{h^2}, & \text{if } 0 \le x \le h, \\ 0, & \text{otherwise.} \end{cases}$$

(*c*) The marginal pmf's of X are obtained from Table 3-1 by computing the row sums, and the marginal pmf's of Y are obtained by computing the column sums. Thus,

$$p_X(0)=\frac{35}{84} \qquad p_X(1)=\frac{42}{84} \qquad p_X(2)=\frac{7}{84}$$

$$p_Y(0)=\frac{20}{84} \qquad p_Y(1)=\frac{45}{84} \qquad p_Y(2)=\frac{18}{84} \qquad p_Y(3)=\frac{1}{84}$$

TABLE 3-1 $p_{XY}(i, j)$

	j			
i	0	1	2	3
0	$\frac{4}{84}$	$\frac{18}{84}$	$\frac{12}{84}$	$\frac{1}{84}$
1	$\frac{12}{84}$	$\frac{24}{84}$	$\frac{6}{84}$	0
2	$\frac{4}{84}$	$\frac{3}{84}$	0	0

(*d*) Since

$$p_{XY}(0,0)=\frac{4}{84} \neq p_X(0)p_Y(0)=\frac{35}{84}\left(\frac{20}{84}\right)$$

X and Y are not independent.

3.14. The joint pmf of a bivariate r.v. (X, Y) is given by

$$p_{XY}(x_i, y_j)=\begin{cases} k(2x_i+y_j) & x_i=1,2;\ y_j=1,2 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

(*a*) Find the value of k.

(*b*) Find the marginal pmf's of X and Y.

(*c*) Are X and Y independent?

(*a*) By Eq. (3.17),

$$\sum_{x_i}\sum_{y_j} p_{XY}(x_i, y_j)=\sum_{x_i=1}^{2}\sum_{y_j=1}^{2} k(2x_i+y_j)$$
$$=k[(2+1)+(2+2)+(4+1)+(4+2)]=k(18)=1$$

Thus, $k=\frac{1}{18}$.

(*b*) By Eq. (3.20), the marginal pmf's of X are

$$p_X(x_i)=\sum_{y_j} p_{XY}(x_i, y_j)=\sum_{y_j=1}^{2}\frac{1}{18}(2x_i+y_j)$$
$$=\frac{1}{18}(2x_i+1)+\frac{1}{18}(2x_i+2)=\frac{1}{18}(4x_i+3) \qquad x_i=1,2$$

By Eq. (3.21), the marginal pmf's of Y are

$$p_Y(y_j) = \sum_{x_i} p_{XY}(x_i, y_j) = \sum_{x_i=1}^{2} \frac{1}{18}(2x_i + y_j)$$
$$= \frac{1}{18}(2 + y_j) + \frac{1}{18}(4 + y_j) = \frac{1}{18}(2y_j + 6) \qquad y_j = 1, 2$$

(*c*) Now $p_X(x_i)p_Y(y_j) \neq p_{XY}(x_i, y_j)$; hence X and Y are not independent.

3.15. The joint pmf of a bivariate r.v. (X, Y) is given by

$$\begin{cases} kx_i^2 y_j & x_i = 1, 2;\ y_j = 1, 2, 3 \end{cases}$$

2.49. Assume that the length of a phone call in minutes is an exponential r.v. X with parameter $\lambda = \frac{1}{10}$. If someone arrives at a phone booth just before you arrive, find the probability that you will have to wait (*a*) less than 5 minutes, and (*b*) between 5 and 10 minutes.

(*a*) From Eq. (2.60), the pdf of X is

$$f_X(x) = \begin{cases} \frac{1}{10} e^{-x/10} & x > 0 \\ 0 & x < 0 \end{cases}$$

Then

$$P(X < 5) = \int_0^5 \frac{1}{10} e^{-x/10}\, dx = -e^{-x/10}\Big|_0^5 = 1 - e^{-0.5} \approx 0.393$$

(*b*) Similarly,

$$P(5 < X < 10) = \int_5^{10} \frac{1}{10} e^{-x/10}\, dx = e^{-0.5} - e^{-1} \approx 0.239$$

2.50. All manufactured devices and machines fail to work sooner or later. Suppose that the failure rate is constant and the time to failure (in hours) is an exponential r.v. X with parameter λ.

(*a*) Measurements show that the probability that the time to failure for computer memory chips in a given class exceeds 10^4 hours is e^{-1} (≈ 0.368). Calculate the value of the parameter λ.

(*b*) Using the value of the parameter λ determined in part (*a*), calculate the time x_0 such that the probability that the time to failure is less than x_0 is 0.05.

```
clear all;
close all;
%%
xy = rand(1,10000);
xy_joint = zeros(2,2);
for i=1:length(xy);
    if xy(i) <= 1/8;
        xy_joint(1,1) = xy_joint(1,1)+1;
    end
    if xy(i) >1/8 && xy(i) <= 2/8;
        xy_joint(1,2) = xy_joint(1,2)+1;
    end
    if xy(i)>2/8 && xy(i)<=1/2;
        xy_joint(2,1) = xy_joint(2,1)+1;
    end
    if xy(i)>1/2 && xy(i) <=1
        xy_joint(2,2) = xy_joint(2,2)+1;
    end
end
xy_joint = xy_joint / 10000
%%
xmarginal = zeros(1,2);
for i=1:length(xy);
    if xy(i) <= 3/8;
        xmarginal(1,1) = xmarginal(1,1)+1;
    elseif xy(i)>3/8 && xy(i) <=1
        xmarginal(1,2) = xmarginal(1,2)+1;
    end
end
xmarginal = xmarginal/10000
%%
ygivenx0conditionalpmf = zeros(1,2);
for i=1:length(xy);
    if xy(i) <= 1/3;
        ygivenx0conditionalpmf(1,1) = ygivenx0conditionalpmf(1,1)+1;
    end
    if xy(i)>1/3 && xy(i) <=1
        ygivenx0conditionalpmf(1,2) = ygivenx0conditionalpmf(1,2)+1;
    end
end
ygivenx0conditionalpmf = ygivenx0conditionalpmf/10000


ygivenx1conditionalpmf = zeros(1,2);
for i=1:length(xy);
    if xy(i) <= 1/5;
        ygivenx1conditionalpmf(1,1) = ygivenx1conditionalpmf(1,1)+1;
    end
    if xy(i)>1/5 && xy(i) <=1
        ygivenx1conditionalpmf(1,2) = ygivenx1conditionalpmf(1,2)+1;
    end
end
ygivenx1conditionalpmf = ygivenx1conditionalpmf/10000

%%Part b
```

```
xy_joint_partb=[ygivenx0conditionalpmf(1)*xmarginal(1) ygivenx1conditionalpmf(1)*xmargina
        ygivenx0conditionalpmf(2)*xmarginal(1) ygivenx1conditionalpmf(2)*xmarginal(2)]

temp = [zeros(1,(xmarginal(1)*10000)),ones(1,(xmarginal(2)*10000))];
%% Part c and Part d
mean(temp)
var(temp)
```