# CHAPTER 3

**Solution to Problem 3.1.** The random variable $Y = g(X)$ is discrete and its PMF is given by

$$p_Y(1) = \mathbf{P}(X \le 1/3) = 1/3, \qquad p_Y(2) = 1 - p_Y(1) = 2/3.$$

Thus,

$$\mathbf{E}[Y] = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 2 = \frac{5}{3}.$$

The same result is obtained using the expected value rule:

$$\mathbf{E}[Y] = \int_0^1 g(x) f_X(x)\,dx = \int_0^{1/3} dx + \int_{1/3}^1 2\,dx = \frac{5}{3}.$$

**Solution to Problem 3.2.** We have

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_{-\infty}^{\infty} \frac{\lambda}{2} e^{-\lambda|x|}\,dx = 2 \cdot \frac{1}{2} \int_0^{\infty} \lambda e^{-\lambda x}\,dx = 2 \cdot \frac{1}{2} = 1,$$

where we have used the fact $\int_0^\infty \lambda e^{-\lambda x} dx = 1$, i.e., the normalization property of the exponential PDF. By symmetry of the PDF, we have $\mathbf{E}[X] = 0$. We also have

$$\mathbf{E}[X^2] = \int_{-\infty}^{\infty} x^2 \frac{\lambda}{2} e^{-\lambda|x|} dx = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx = \frac{2}{\lambda^2},$$

where we have used the fact that the second moment of the exponential PDF is $2/\lambda^2$. Thus

$$\text{var}(X) = \mathbf{E}[X^2] - \big(\mathbf{E}[X]\big)^2 = 2/\lambda^2.$$

**Solution to Problem 3.5.** Let $A = bh/2$ be the area of the given triangle, where $b$ is the length of the base, and $h$ is the height of the triangle. From the randomly chosen point, draw a line parallel to the base, and let $A_x$ be the area of the triangle thus formed. The height of this triangle is $h - x$ and its base has length $b(h - x)/h$. Thus $A_x = b(h - x)^2/(2h)$. For $x \in [0, h]$, we have

$$F_X(x) = 1 - \mathbf{P}(X > x) = 1 - \frac{A_x}{A} = 1 - \frac{b(h-x)^2/(2h)}{bh/2} = 1 - \left(\frac{h-x}{h}\right)^2,$$

while $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x > h$.

The PDF is obtained by differentiating the CDF. We have

$$f_X(x) = \frac{dF_X}{dx}(x) = \begin{cases} \dfrac{2(h-x)}{h^2}, & \text{if } 0 \le x \le h, \\ 0, & \text{otherwise.} \end{cases}$$

**Solution to Problem 3.6.** Let $X$ be the waiting time and $Y$ be the number of customers found. For $x < 0$, we have $F_X(x) = 0$, while for $x \geq 0$,

$$F_X(x) = \mathbf{P}(X \leq x) = \frac{1}{2}\mathbf{P}(X \leq x \,|\, Y = 0) + \frac{1}{2}\mathbf{P}(X \leq x \,|\, Y = 1).$$

Since

$$\mathbf{P}(X \leq x \,|\, Y = 0) = 1,$$

$$\mathbf{P}(X \leq x \,|\, Y = 1) = 1 - e^{-\lambda x},$$

we obtain

$$F_X(x) = \begin{cases} \frac{1}{2}(2 - e^{-\lambda x}), & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the CDF has a discontinuity at $x = 0$. The random variable $X$ is neither discrete nor continuous.

**Solution to Problem 3.7.** (a) We first calculate the CDF of $X$. For $x \in [0, r]$, we have

$$F_X(x) = \mathbf{P}(X \leq x) = \frac{\pi x^2}{\pi r^2} = \left(\frac{x}{r}\right)^2.$$

For $x < 0$, we have $F_X(x) = 0$, and for $x > r$, we have $F_X(x) = 1$. By differentiating, we obtain the PDF

$$f_X(x) = \begin{cases} \frac{2x}{r^2}, & \text{if } 0 \leq x \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\mathbf{E}[X] = \int_0^r \frac{2x^2}{r^2}\,dx = \frac{2r}{3}.$$

Also

$$\mathbf{E}[X^2] = \int_0^r \frac{2x^3}{r^2}\,dx = \frac{r^2}{2},$$

so

$$\text{var}(X) = \mathbf{E}[X^2] - \big(\mathbf{E}[X]\big)^2 = \frac{r^2}{2} - \frac{4r^2}{9} = \frac{r^2}{18}.$$

(b) Alvin gets a positive score in the range $[1/t, \infty)$ if and only if $X \leq t$, and otherwise he gets a score of 0. Thus, for $s < 0$, the CDF of $S$ is $F_S(s) = 0$. For $0 \leq s < 1/t$, we have

$$F_S(s) = \mathbf{P}(S \leq s) = \mathbf{P}(\text{Alvin's hit is outside the inner circle}) = 1 - \mathbf{P}(X \leq t) = 1 - \frac{t^2}{r^2}.$$

For $1/t < s$, the CDF of $S$ is given by

$$F_S(s) = \mathbf{P}(S \leq s) = \mathbf{P}(X \leq t)\mathbf{P}(S \leq s \,|\, X \leq t) + \mathbf{P}(X > t)\mathbf{P}(S \leq s \,|\, X > t).$$

$$F_Z(z) = \begin{cases} 0, & \text{if } z < 0, \\ 1 - e^{-\lambda z}, & \text{if } z \geq 0. \end{cases}$$

We have $f_X(x) = pf_Y(x) + (1-p)f_Z(x)$, and consequently $F_X(x) = pF_Y(x) + (1-p)F_Z(x)$. It follows that

$$F_X(x) = \begin{cases} pe^{\lambda x}, & \text{if } x < 0, \\ p + (1-p)(1 - e^{-\lambda x}), & \text{if } x \geq 0, \end{cases}$$
$$= \begin{cases} pe^{\lambda x}, & \text{if } x < 0, \\ 1 - (1-p)e^{-\lambda x}, & \text{if } x \geq 0. \end{cases}$$

**Solution to Problem 3.11.** (a) $X$ is a standard normal, so by using the normal table, we have $\mathbf{P}(X \leq 1.5) = \Phi(1.5) = 0.9332$. Also $\mathbf{P}(X \leq -1) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$.

(b) The random variable $(Y-1)/2$ is obtained by subtracting from $Y$ its mean (which is 1) and dividing by the standard deviation (which is 2), so the PDF of $(Y-1)/2$ is the standard normal.

(c) We have, using the normal table,

$$\begin{aligned} \mathbf{P}(-1 \leq Y \leq 1) &= \mathbf{P}\big(-1 \leq (Y-1)/2 \leq 0\big) \\ &= \mathbf{P}(-1 \leq Z \leq 0) \\ &= \mathbf{P}(0 \leq Z \leq 1) \\ &= \Phi(1) - \Phi(0) \\ &= 0.8413 - 0.5 \\ &= 0.3413, \end{aligned}$$

where $Z$ is a standard normal random variable.

**Solution to Problem 3.12.** The random variable $Z = X/\sigma$ is a standard normal, so

$$\mathbf{P}(X \geq k\sigma) = \mathbf{P}(Z \geq k) = 1 - \Phi(k).$$

From the normal tables we have

$$\Phi(1) = 0.8413, \qquad \Phi(2) = 0.9772, \qquad \Phi(3) = 0.9986.$$

Thus $\mathbf{P}(X \geq \sigma) = 0.1587$, $\mathbf{P}(X \geq 2\sigma) = 0.0228$, $\mathbf{P}(X \geq 3\sigma) = 0.0014$.

We also have

$$\mathbf{P}\big(|X| \leq k\sigma\big) = \mathbf{P}\big(|Z| \leq k\big) = \Phi(k) - \mathbf{P}(Z \leq -k) = \Phi(k) - \big(1 - \Phi(k)\big) = 2\Phi(k) - 1.$$

Using the normal table values above, we obtain

$$\mathbf{P}(|X| \leq \sigma) = 0.6826, \qquad \mathbf{P}(|X| \leq 2\sigma) = 0.9544, \qquad \mathbf{P}(|X| \leq 3\sigma) = 0.9972,$$

where $t$ is a standard normal random variable.

**Solution to Problem 3.13.** Let $X$ and $Y$ be the temperature in Celsius and Fahrenheit, respectively, which are related by $X = 5(Y - 32)/9$. Therefore, 59 degrees Fahrenheit correspond to 15 degrees Celsius. So, if $Z$ is a standard normal random variable, we have using $\mathbf{E}[X] = \sigma_X = 10$,

$$\mathbf{P}(Y \leq 59) = \mathbf{P}(X \leq 15) = \mathbf{P}\left(Z \leq \frac{15 - \mathbf{E}[X]}{\sigma_X}\right) = \mathbf{P}(Z \leq 0.5) = \Phi(0.5).$$

From the normal tables we have $\Phi(0.5) = 0.6915$, so $\mathbf{P}(Y \leq 59) = 0.6915$.

**Solution to Problem 3.15.** (a) Since the area of the semicircle is $\pi r^2/2$, the joint PDF of $X$ and $Y$ is $f_{X,Y}(x, y) = 2/\pi r^2$, for $(x, y)$ in the semicircle, and $f_{X,Y}(x, y) = 0$, otherwise.

(b) To find the marginal PDF of $Y$, we integrate the joint PDF over the range of $X$. For any possible value $y$ of $Y$, the range of possible values of $X$ is the interval $[-\sqrt{r^2 - y^2}, \sqrt{r^2 - y^2}]$, and we have

$$f_Y(y) = \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{2}{\pi r^2}\, dx = \begin{cases} \dfrac{4\sqrt{r^2 - y^2}}{\pi r^2}, & \text{if } 0 \leq y \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$\mathbf{E}[Y] = \frac{4}{\pi r^2} \int_0^r y\sqrt{r^2 - y^2}\, dy = \frac{4r}{3\pi},$$

where the integration is performed using the substitution $z = r^2 - y^2$.

(c) There is no need to find the marginal PDF $f_Y$ in order to find $\mathbf{E}[Y]$. Let $D$ denote the semicircle. We have, using polar coordinates

$$\mathbf{E}[Y] = \int\int_{(x,y)\in D} y f_{X,Y}(x, y)\, dx\, dy = \int_0^\pi \int_0^r \frac{2}{\pi r^2}\, s(\sin\theta)s\, ds\, d\theta = \frac{4r}{3\pi}.$$

**Solution to Problem 3.16.** Let $A$ be the event that the needle will cross a horizontal line, and let $B$ be the probability that it will cross a vertical line. From the analysis of Example 3.11, we have that

$$\mathbf{P}(A) = \frac{2l}{\pi a}, \qquad \mathbf{P}(B) = \frac{2l}{\pi b}.$$

Since at most one horizontal (or vertical) line can be crossed, the expected number of horizontal lines crossed is $\mathbf{P}(A)$ [or $\mathbf{P}(B)$, respectively]. Thus the expected number of crossed lines is

$$\mathbf{P}(A) + \mathbf{P}(B) = \frac{2l}{\pi a} + \frac{2l}{\pi b} = \frac{2l(a + b)}{\pi ab}.$$

The probability that at least one line will be crossed is

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B).$$

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%%
%4 problem 3.7
%  Generate values from the uniform distribution on the interval [a, b].
%            r = a + (b-a).*rand(100,1);
N = 10000
x = [];
point11=(2)* rand([2 N]);
point =  -1+point11;
for i=1:N
    temp = sqrt(point(1,i).^2+point(2,i).^2);
    if temp < 1
        x = [x temp];

    end
end
figure(1);
subplot(1,2,1);
hist(x);
subplot(1,2,2);
[n xout] = hist(x,50);
area = (xout(2)-xout(1))*sum(n);
bar(xout, n/area);

Expectation = mean(x)
Variance = var(x)
%%
%%
%6 problem 3.10
N = 10000;
x = rand(1,N);
lambda = 2;
value = -log(1-x)/lambda;
figure(2);

subplot(1,2,1);
hist(value);
subplot(1,2,2);
[n xout] = hist(value,50);
area = (xout(2)-xout(1))*sum(n);
bar(xout, n/area);
```

By Eq. (2.24), the cdf of $X$ is given by

$$F_X(x)=\begin{cases}0 & x<0\\ \int_0^x \frac{1}{3}\,d\xi=\frac{x}{3} & 0\le x<1\\ \int_0^1 \frac{1}{3}\,d\xi+\int_1^x \frac{2}{3}\,d\xi=\frac{2}{3}x-\frac{1}{3} & 1\le x<2\\ \int_0^1 \frac{1}{3}\,d\xi+\int_1^2 \frac{2}{3}\,d\xi=1 & 2\le x\end{cases}$$

$$= \frac{1}{3}+\int_1^x \frac{2}{3}\,dx \qquad \frac{1}{3}+\frac{2}{3}(x-1)$$

The functions $f_X(x)$ and $F_X(x)$ are sketched in Fig. 2-21.

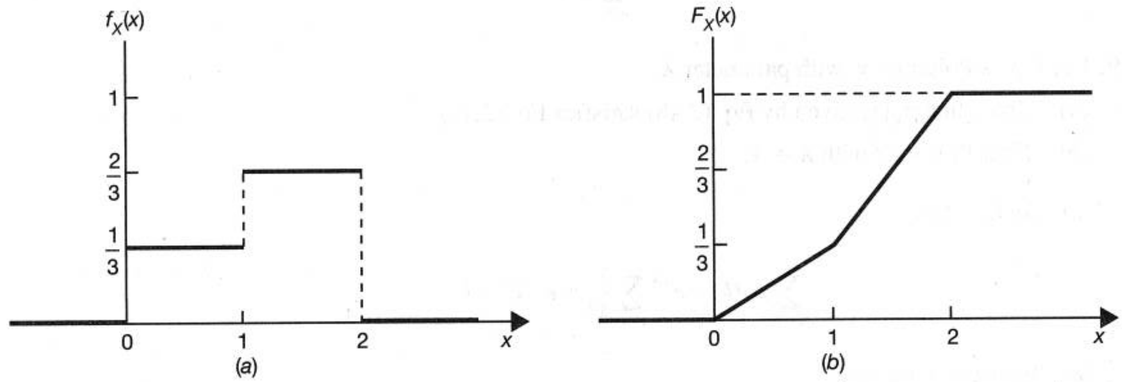


Fig. 2-21

**2.22.** Let $X$ be a continuous r.v. $X$ with pdf

$$f_X(x)=\begin{cases}kx & 0<x<1\\ 0 & \text{otherwise}\end{cases}$$

where $k$ is a constant.

(*a*) Determine the value of $k$ and sketch $f_X(x)$.

(*b*) Find and sketch the corresponding cdf $F_X(x)$.

(*c*) Find $P(\frac{1}{4}<X\le 2)$.

(*a*) By Eq. (2.21), we must have $k>0$, and by Eq. (2.22),

$$\int_0^1 kx\,dx=\frac{k}{2}=1$$

Thus, $k=2$ and

$$f_X(x)=\begin{cases}2x & 0<x<1\\ 0 & \text{otherwise}\end{cases}$$

which is sketched in Fig. 2-22(*a*).