ECEn 483 / ME 431 Case Studies

Randal W. Beard
Brigham Young University

Updated: November 29, 2013

Contents

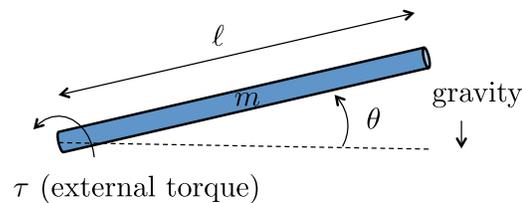# Design Problem 1

Single Link Robot Arm

Figure 1.1: Single Link Robot Arm

Figure 1.1 shows the single link robot arm system. The robot arm has mass m and length ℓ. The angle of the robot is given by θ as measured from level. The angular speed of the arm is $\dot{\theta}$. There is an applied torque τ at the joint, and assuming a damping torque that opposes the rotation of the joint of magnitude $-b\dot{\theta}$.

Assume the following physical constants: $m = 0.5$ kg, $\ell = 0.3$ m, $g = 9.8$ m/s^2, $b = 0.01$ Nms.

Homework I.1 Using the configuration variable θ, write an expression for the kinetic energy of the system.

Homework I.2 Create a simulink animation of the single link robot arm. The input should be a slider for θ.

Homework I.3

(a) Find the potential energy for the system.

1

(b) Define the generalized coordinates.

(c) Find the generalized forces.

(d) Derive the equations of motion using the Euler-Lagrange equations.

Homework I.4 Modify the simulink model created in homework I.2 by creating an s-function that implements the equations of motion. The input to the s-function should be a slider for torque. The output should go to the animation developed in homework I.2.

Homework I.5

(a) Find the equilibria of the system.

(b) Linearize the system about the equilibria.

Homework I.6 Find the transfer function of the system from the torque τ to the angle θ.

Homework I.7 Defining the states as $x = (\tilde{\theta}, \dot{\tilde{\theta}})^\top$, the input as $u = \tilde{\tau}$, and the measured output as $y = \tilde{\theta}$, find the linear state space equations in the form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

Homework I.8 Suppose that the size of the input torque is limited to $\tau_{\max} = 1$ Nm.

(a) Suppose that a PD controller is used to stabilize the system and regulate θ to θ^d. Draw the (linearized) closed loop block diagram using the transfer function found in problem I.6. Derive the actual transfer function from $\tilde{\theta}^d$ to $\tilde{\theta}$.

(b) Considering the equilibrium torque τ_e, derive a bound $\tau_{\max}$ on the linearized torque $\tilde{\tau}$ such that $|\tilde{\tau}| \leq \tau_{\max}$ when the maximum possible equilibrium torque is used.

(c) Select the proportional gain k_p so that $\tilde{\tau}$ just saturates when a step of size $A_{th} = 50$ degrees is placed on $\tilde{\theta}_d$.

(d) If the desired transfer function is given by

$$\tilde{\theta}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\tilde{\theta}^d(s),$$

find the natural frequency ω_n and the proportional gain k_d so that the actual transfer function equals the desired transfer function, where ζ is a design parameter.

(e) What are the closed loop poles?

Homework I.9 Implement the PD controller designed in Homework I.8 on the single link robot arm in Simulink. Note that the input torque from the PD controller is given by

$$\tilde{\tau} = k_p(\theta^d - \theta) - k_d\dot{\theta}.$$

You should select the input torque as

$$\tau = \tau_e + \tilde{\tau},$$

where τ_e is the equilibrium torque found in Homework I.5.

Homework I.10

(a) With PD control, what is the system type? Characterize the steady state error when the reference input is a step, a ramp, and a parabola. How does this change if you add an integrator?

(b) Consider the case where a constant disturbance acts at the input to the plant (for example gravity in this case). What is the steady state error to a constant input disturbance when the integrator is not present, and when it is present?

(c) Add an integrator to the PD control developed in Homework I.9. Draw the block diagram of the closed loop system with PID control. Find the transfer function for the closed loop system with the integrator added.

(d) Put the characteristic equation in Evan's form and use the Matlab SISO tool to plot the root locus verses the integrator gain k_i.

Homework I.11 The objective of this problem is to implement the PID controller using Matlab code using only measured outputs of the system.

(a) Change the parameter file param.m to specify actual parameters and the parameters known to the controller. Change the values for mass and length known to the controller by 5% to simulate imperfect knowledge of the plant.

(b) Rearrange the block diagram so that the controller is implemented as an m-function implemented at the sample rate of $T_s = 0.01$. The controller should only assume knowledge of the angle θ and the desired angle θ^d.

(c) Implement the PID controller designed in Problems I.8 and I.10 using an m-function called arm_ctrl.m. Use the dirty derivative gain of $\tau = 0.05$.

Homework I.12 Draw by hand the Bode plot of the single link robot arm from torque $\tilde{\tau}$ to angle $\tilde{\theta}$ given that the equilibrium angle is $\theta_e = 0$. Use the Matlab `bode` command and compare your results.

Homework I.13 For the single link robot arm, use the Matlab `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, (2) the plant under PD control, and (3) the plant under PID control, using the control gains calculated in Homework 11.

(a) To what percent error can the closed loop system under PID control track the desired input if all of the frequency content of $\theta^d(t)$ is below $\omega = 0.4$ radians per second?

(b) If the desired input is $\theta^d(t) = 5t$ for $t \geq 0$, what will be the steady state tracking error to this input?

(c) If all of the frequency content of the input disturbance $d_i(t)$ is below $\omega = 0.01$ radians per second, what percentage of the input disturbance shows up in the output θ?

(d) If all of the frequency content of the noise $n(t)$ is greater than $\omega = 100$ radians per second, what percentage of the noise shows up in the output signal θ?

Homework I.14 For this homework assignment we will use loopshaping to improve the PD controllers developed in Homework 9. Download the simulink files on the web that are associated with this homework. When you open the Simulink file, notice the presence of the LTI system with parameter C, and also the prefilter with parameter F.

(a) At the Matlab prompt, type `>> C = 1` and `F=1` to set the additional compensator ant prefilter to one. Run the simulation to establish baseline performance.

(b) Note in `param.m` the line `L=series(G,C_pd)`, where $L(s)$ is the loop gain. Load the `sisotool` and enter L for the system plant under the tab `System Data`.

(c) The objective will be to (1) improve tracking and disturbance rejection by a factor of 10 for reference signals and disturbances below 0.007 radians/sec, (2) improve noise attention by a factor of 10 for frequencies above 200 radians/sec. In `sisotool` add constraints on the open loop Bode plot that reflect these objectives.

(d) Add a phase lag filter to satisfy the disturbance rejection and tracking objective, and add one to two poles to meet the noise rejection objective. If needed, add a phase lead filter to keep the phase margin around 60 degrees.

(e) Export $C(s)$, and run the combined controller in Simulink.

(f) Note that despite having a good phase margin, there is still significant overshoot, due in part to the windup effect in the phase lag filter. This can be mitigated by adding a prefilter, that essentially modifies the hard step input into the system. In sisotool, add the a plot for the closed loop Bode response. Add a loop pass filter for $F(s)$ as a prefilter to flatten the closed loop Bode response.

Homework I.15 The objective of this problem is to implement state feedback controller using the full state. Start with the simulation files developed in Homework I.11.

(a) Modify arm_dynamics.m so that the outputs are $(\theta, x)^\top$, where $x = (\theta, \dot{\theta})^\top$. In this problem we will use x instead of θ but in the next problem we will again remove x as an output of the plant and reconstruct the state using an observer.

(b) Rearrange the block diagram so that the state x is an input to the controller.

(c) Select the closed loop poles as the roots of the equations $s^2 + 2\zeta\omega_n + \omega_n^2 = 0$ where ω_n, and ζ were found in Homework I.11.

(d) Add the state space matrices A, B, C, D derived in Homework I.7 to your param file.

(e) Verify that the state space system is controllable by checking that rank$(\mathcal{C}) = n$.

(f) Using the Matlab `place` command, find the feedback gain K such that the eigenvalues of $(A - BK)$ are equal to desired closed loop poles. Find the reference gain k_r so that the DC-gain from θ^c to θ is equal to one. Note that $K = (k_p, k_d)$ where k_p and k_d are the proportional and derivative gains found in Homework I.11. Why?

(g) Implement the state feedback scheme in Simulink and tune the closed loop poles to get good response.

Hints:

- For comparison, using exact parameters, I get the following state space model:

$$A = \begin{pmatrix} 0 & 1.0000 \\ 0 & -0.667 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 66.667 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

- We also need to input the equilibrium torque as

$$\tau_e = \frac{mg\ell}{2}\cos(\theta^d)$$

$$\tau = \tau_e - Kx + k_r\theta^d.$$

Homework I.16

(a) Modify the state feedback solution developed in Homework 15 to add an integrator with anti-windup.

(b) Change the parameters known to the controller by $\pm 5\%$.

(c) Tune the integrator to get good tracking performance.

Homework I.17 The objective of this problem is to replace the state x in Homework 16 with the estimated state $\hat{x}$ produced by an observer.

(a) Modify the simulink diagram from Homework 16 so that y is also an input to the controller, as well as x, and so that $\hat{x}$ is also an output of the controller. Add a scope to plot the estimation error $x - \hat{x}$.

(b) Verify that the state space system is observable by checking that $\text{rank}(\mathcal{O}) = n$.

(c) In the control block, add an observer to estimate the state $\hat{x}$. Do not yet use $\hat{x}$ in the state feedback equations. Run the controller using the actual state x and tune the observer so that the estimation error is small. Make sure the $\text{eig}(A - BK - LC)$ are in the left half plane.

(d) Replace x in the controller with the estimated state $\hat{x}$ and tune the controller and observer if necessary to obtain good performance.

(e) Modify the simulink diagram to add an input disturbance of 0.2 Newton-meters. Add a state to the observer to estimate the disturbance. Replace the integrator with an estimate of the disturbance and compare the performance.

Hints:

- *Note that the observer is estimating the linearized state $\tilde{x} = x - x_e$. Therefore the observer update equation will be*

$$\dot{\hat{\tilde{x}}} = A\hat{\tilde{x}} + B\tilde{u} + L(\tilde{y} - C\hat{\tilde{x}}),$$

but sense $\dot{\tilde{x}} = \dot{x} - \dot{x}_e = \dot{x}$, and since $y_e = Cx_e$, this is equivalent to

$$\dot{\hat{x}} = A(\hat{x} - x_e) + B(u - u_e) + L(y - C\hat{x}).$$
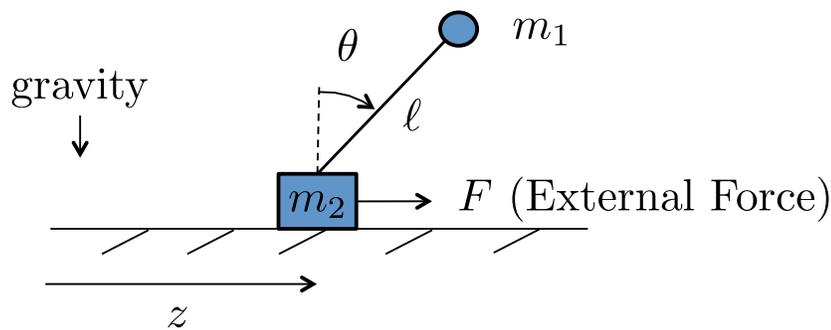
Design Problem 2

Pendulum on a Cart

Figure 2.1: Pendulum on a cart.

Figure 2.1 shows the pendulum on a cart system. The position of the cart measured from the origin is z and the linear speed of the cart is $\dot{z}$. The angle of the pendulum from straight up is given by θ and the angular velocity is $\dot{\theta}$. The pendulum arm is assumed to be massless, and m_1 is assumed to be a point mass. Gravity acts in the down direction. The only applied force is F which acts in the direction of z. The cart slides on a frictionless surface but air friction produces a damping force equal to $-b\dot{z}$.

Assume the following physical constants: $m_1 = 0.25$ kg, $m_2 = 1.0$ kg, $\ell = 0.5$ m, $g = 9.8$ m/s^2, $b = 0.05$ Ns.

Homework II.1 Using the configuration variables z and θ, write an expression for the kinetic energy of the system.

Homework II.2 Create a simulink animation of the pendulum on a cart
system. The inputs should be sliders for z and θ.

Homework II.3

   (a) Find the potential energy for the system.

   (b) Define the generalized coordinates.

   (c) Find the generalized forces.

   (d) Derive the equations of motion using the Euler-Lagrange equations.

Homework II.4 Modify the simulink model created in homework II.2 by
creating an s-function that implements the equations of motion. The
input to the s-function should be a slider for force. The output should
go to the animation developed in homework II.2.

Homework II.5

   (a) Find the equilibria of the system.

   (b) Linearize the system about the equilibria.

Homework II.6

   (a) Using the Laplace transform, convert from time domain to the
       s-domain.

   (b) Find the transfer functions that define the cascade connected from
       force F to the cart position z to the pendulum angle θ.

   (c) Find explicit expressions for the disturbances that have been ignored.

Homework II.7 Defining the states as $x = (z, \tilde{\theta}, \dot{z}, \dot{\tilde{\theta}})^\top$, the input as $u = F$,
and the measured output as $y = (z, \tilde{\theta})^\top$, find the linear state space
equations in the form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

Homework II.8 Suppose that the size of the input force on the cart is
limited to $F_{\max} = 5$ N. The objective of this problem is to design
separate PD controllers for the inner loop (force to angle) and the
outer loop (angle to position).

(a) Using the transfer functions derived inn problem II.6, and using a PD controller for the inner loop to regulate $\tilde{\theta}$ to $\tilde{\theta}^d$, and a PD controller for the outer loop to regulate $\tilde{z}$ to ^{d}, draw the block diagram of the complete closed loop system.

(b) Derive the actual transfer function of the inner loop from $\tilde{\theta}^d$ to $\tilde{\theta}$.

(c) Considering the equilibrium force F_e, derive a bound $F_{\max}$ on the linearized force $\tilde{F}$ such that $\left|\tilde{F}\right| \leq F_{\max}$ when the maximum possible equilibrium force is used.

(d) Select the proportional gain k_{p_θ} so that $\tilde{F}$ just saturates when a step of size A_{th} is placed on $\tilde{\theta}_d$.

(e) If the desired transfer function is given by

$$\tilde{\theta}(s) = \frac{\omega_{n_\theta}^2}{s^2 + 2\zeta_\theta\omega_{n_\theta}s + \omega_{n_\theta}^2}\tilde{\theta}^d(s),$$

find the natural frequency ω_{n_θ} and the derivative gain k_{d_θ} so that the actual transfer function equals the desired transfer function, where ζ_θ is a design parameter.

(f) What are the closed loop poles of the inner loop?

(g) Assuming that the inner loop is working properly and that $\tilde{\theta} \approx \tilde{\theta}^d$, draw the block diagram for the outer loop and derive the actual transfer function from $\tilde{z}^d$ to $\tilde{z}$.

(h) Select the proportional gain k_{p_z} so that $\tilde{\theta}^d$ just saturates at A_{th} when a step of size $A_z = 2$ meters is placed on $\tilde{z}^d$.

(i) If the desired transfer function is given by

$$\tilde{z}(s) = \frac{\omega_{n_z}^2}{s^2 + 2\zeta_z\omega_{n_z}s + \omega_{n_z}^2}\tilde{z}^d(s),$$

find the natural frequency ω_{n_z} and the derivative gain k_{d_z} so that the actual transfer function equals the desired transfer function, where ζ_z is a design parameter.

(j) What are the closed loop poles of the outer loop?

Homework II.9 Implement the PD controllers for the inner and outer loops designed in Homework II.8 on the inverted in Simulink.

Homework II.10

(a) Consider the inner loop with PD control, what is the system type? Characterize the steady state error when $\tilde{\theta}^d$ is a step, a ramp, and a parabola. How does this change if you add an integrator?

(b) Consider the outer loop with PD control, what is the system type? Characterize the steady state error when $\tilde{z}^d$ is a step, a ramp, and a parabola. How does this change if you add an integrator?

Homework II.11 The objective of this problem is to implement the PID controller using Matlab code using only measured outputs of the system.

(a) Change the parameter file param.m to specify actual parameters and the parameters known to the controller. Change the values for m_1, m_2, ℓ and b known to the controller by 5% to simulate imperfect knowledge of the plant.

(b) Rearrange the block diagram so that the controller is implemented as an m-function implemented at the sample rate of $T_s = 0.01$. The controller should only assume knowledge of the position z and the angle θ, as well as the desired position z^d.

(e) Implement the nested PID loops designed in Problems II.8 and II.10 using an m-function called pendulum_ctrl.m. Use the dirty derivative gain of $\tau = 0.05$.

Homework II.12

(a) Draw by hand the Bode plot of the inner loop transfer function from force F to angle θ for the inverted pendulum. Use the Matlab `bode` command and compare your results.

(b) Draw by hand the Bode plot of the outer loop transfer function from force F to angle θ for the inverted pendulum. Use the Matlab `bode` command and compare your results.

Homework II.13 For the inner loop of the inverted pendulum, use the Matlab `bode` command to create a graph that simultaneously displays

the Bode plots for (1) the plant, (2) the plant under PD control, and (3) the plant under PID control, using the control gains calculated in Homework 11.

(a) To what percent error can the closed loop system under PID control track the desired input if all of the frequency content of $\theta^d(t)$ is below $\omega = 1.0$ radians per second?

(b) If the commaned input to the inner loop $\theta^d(t)$ is a step of size 2.0 radians what will be the steady state tracking error under PD control?

(c) If all of the frequency content of the noise $n(t)$ is greater than $\omega = 200$ radians per second, what percentage of the noise shows up in the output signal θ?

For the outer loop of the inverted pendulum, use the Matlab `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, and (2) the plant under PD control, using the control gains calculated in Homework 11.

(d) If the frequency content of an output disturbance is contained below 0.1 radian/sec, what percentage of the output disturbance will be contained in the output?

(e) If the reference signal $y^d(t)$ is a ramp of magnitude 2 meters/sec, what is the tracking error.

Homework II.14 For this homework assignment we will use loopshaping to improve the PD controllers developed in Homework 9. Download the simulink files on the web that are associated with this homework. We will focus on the outer loop. When you open the Simulink file, notice the presence of the LTI system with parameter C.

(a) At the Matlab prompt, type `>> C = 1` to set the additional compensator originally to one. Run the simulation to establish baseline performance.

(b) Note in `param.m` the line `L=series(P_out,C_out_pd)`, where $L(s)$ is the loop gain for the out loop. Load the `sisotool` and enter L for the system plant under the tab `System Data`.

(c) The objective will be to (1) improve disturbance rejection and tracking by a factor of 10 for input signals and disturbance below 0.001 radians/sec, (2) improve noise attention by a factor of 10 for frequencies above 20 radians/sec. In `sisotool` add constraints that reflect these objectives.

(d) Add a phase lag filter to satisfy the disturbance rejection and tracking objective, and add one to two poles to meet the noise rejection objective. The phase margin in this case is interpreted different. The phase above 0 degrees corresponds to the normal phase margin. This should remain around 60 degrees. To understand why, you will need to understand the Nyquist criteria.

(e) Export $C(s)$, and run the combined controller in Simulink.

Homework II.15 The objective of this problem is to implement state feedback controller using the full state. Start with the simulation files developed in Homework II.11.

(a) Modify pendulum_dynamics.m so that the outputs are $(z, \theta, x)^\top$, where $x = (z, \theta, \dot{z}, \dot{\theta})^\top$. In this problem we will use x instead of z and θ, but in the next problem we will again remove x as an output of the plant and reconstruct the state using an observer.

(b) Rearrange the block diagram so that the state x is an input to the controller.

(c) Using the values for ω_{n_z}, ζ_z, ω_{n_θ}, and ζ_θ selected in Homework 11, find the desired closed loop poles.

(d) Add the state space matrices A, B, C, D derived in Homework 7 to your param file.

(e) Verify that the state space system is controllable by checking that $\text{rank}(\mathcal{C}) = n$.

(f) Using the Matlab `place` command, find the feedback gain K such that the eigenvalues of $(A - BK)$ are equal to desired closed loop poles. Find the reference gain k_r so that the DC-gain from z^c to z is equal to one.

(g) Implement the state feedback scheme in Simulink and tune the closed loop poles to get good response. You should be able to get much faster response using state space methods.

Hints:

- There are several mistakes in previous solutions. For comparison, using exact parameters, I get the following state space model:

$$A = \begin{pmatrix} 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0 & -2.4500 & -0.0500 & 0 \\ 0 & 24.5000 & 0.1000 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- Be careful in computing $k_r = -1/C(A-BK)^{-1}B$. In this formula, C is the output matrix matching the reference input. Therefore, for this problem $C = (1, 0, 0, 0)$.

Homework II.16

(a) Modify the state feedback solution developed in Homework 15 to add an integrator with anti-windup to the position feedback.

(b) Change the parameters known to the controller by $\pm 5\%$.

(c) Add a constant input disturbance of 0.5 Newtons to the input of the plant and verify that when the integrator gain is set to zero, there is a steady state tracking error.

(c) Tune the integrator to get zero steady state tracking performance.

Hints:

- The integrator gain will be negative.

- To get fast integrator action, try turning off the integrator during transition, i.e., when $|\dot{y}|$ is greater than some limit.

Homework II.17 The objective of this problem is to replace the state in Homework 16 with the estimated state $\hat{x}$ produced by an observer.

(a) Modify the simulink diagram from Homework 16 so that y is also an input to the controller, as well as x, and so that $\hat{x}$ is also an output of the controller. Add a scope to plot the estimation error $x - \hat{x}$.

(b) Verify that the state space system is observable by checking that $\mathrm{rank}(\mathcal{O}) = n$.

(c) In the control block, add an observer to estimate the state $\hat{x}$. Do not yet use $\hat{x}$ in the state feedback equations. Run the controller using the actual state x and tune the observer so that the estimation error is small. Make sure the $\mathrm{eig}(A - BK - LC)$ are in the left half plane.

(d) Replace x in the controller with the estimated state $\hat{x}$ and tune the controller and observer if necessary to obtain good performance.

(e) Modify the simulink diagram to add an input disturbance of 0.2 Newton-meters. Add a state to the observer to estimate the input disturbance . Replace the integrator with an estimate of the disturbance and compare the performance.

Hints:

- *Note the eigenvalues of the controller: $\mathrm{eig}(A - BK - LC)$. If any of the eigenvalues of the observer are in the RHP, then tune the desired poles if $\mathrm{eig}(A - LC)$ until all of $\mathrm{eig}(A - BK - LC)$ are in the LHP.*

Design Problem 3

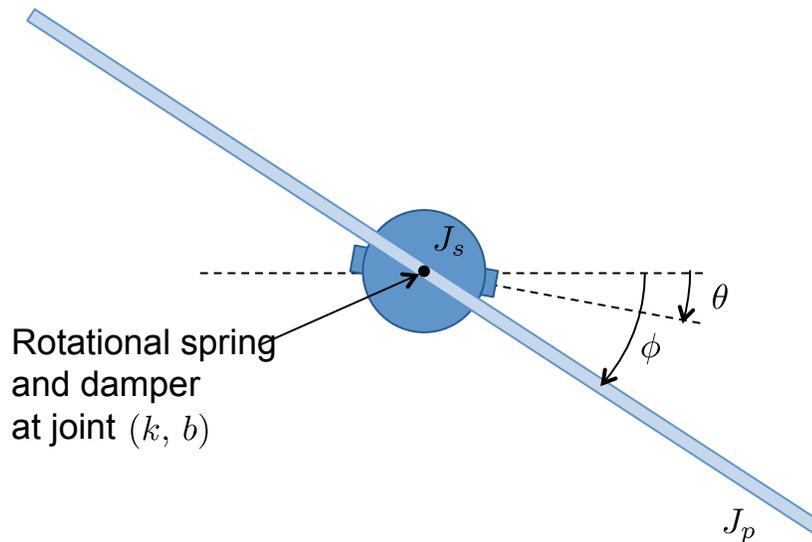# Satellite Attitude Control

Figure 3.1: Satellite with flexible solar panels.

Figure 3.1 shows a simplified version of a satellite with flexible solar panels. We will model the flexible panels using a rigid sheet with moment of inertia J_p connected to the main satellite body by a torsional compliant element with spring constant k and damping constant b. The moment of inertia of the satellite J_s. The angle of satellite body from the inertial reference is θ and the angle of the panel from the inertial reference is denoted as ϕ. Thrusters are used to command an external torque of τ about the axis of the satellite body.

Assume the following physical constants: $J_s = 5$ kg m^2, $J_p = 1$ kg m^2, $k = 0.15$ N m, $b = 0.05$ Nms.

Homework III.1 Using the configuration variables θ and ϕ, write an expression for the kinetic energy of the system.

Homework III.2 Create a simulink animation of the satellite system. The inputs should be sliders for θ and ϕ.

Homework III.3

(a) Find the potential energy for the system.

(b) Define the generalized coordinates.

(c) Find the generalized forces.

(d) Derive the equations of motion using the Euler-Lagrange equations.

(e) Express the equations of motion as four first-order differential equations with states $x = (\theta, \phi, \dot{\theta}, \dot{\phi})^\top$.

Homework III.4 Modify the simulink model created in homework III.2 by creating an s-function that implements the equations of motion. The input to the s-function should be a slider for torque. The output should go to the animation developed in homework III.2.

Homework III.5 The system is linear and it doesn't make sense to linearize.

Homework III.6

(a) Using the Laplace transform, convert from time domain to the s-domain.

(b) Find the transfer functions that define the cascade connected from the torque τ to the angle θ, and the transfer function from θ to ϕ.

(c) Find explicit expressions for the disturbances that have been ignored.

Homework III.7 Suppose that a star tracker is used to measure θ and a strain gage is used to approximate $\phi - \theta$. Defining the states as

$x = (\theta, \phi, \dot{\theta}, \dot{\phi})^\top$, the input as $u = \tau$, and the measured output as $y = (\theta, \phi - \theta)^\top$, find the linear state space equations in the form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

Homework III.8 Suppose that the size of the input torque on the satellite body is limited to $\tau_{\max} = 5$ Nm. Design a PD controller for the inner loop and a proportional (P) controller for the outer loop.

(a) Using the transfer functions derived inn problem III.6, and using a PD controller for the inner loop to regulate θ to θ^d, and a P controller for the outer loop to regulate ϕ to ϕ^d, draw the block diagram of the complete closed loop system. Note that only a P controller is needed on the outer loop because the outer loop is a first order system.

(b) Using $\tau_{\max}$ find the proportional gain k_{p_θ} so that the torque τ is equal to $\tau_{\max}$ the moment after a step size of A_θ is placed on θ^d.

(c) Derive the actual transfer function of the inner loop from θ^d to θ.

(d) If the desired transfer function is given by

$$\theta(s) = \frac{\omega_{n\theta}^2}{s^2 + 2\zeta_\theta \omega_{n\theta} s + \omega_{n\theta}^2} \theta^d(s),$$

find the natural frequency $\omega_{n\theta}$ and the derivative gain k_{d_θ} so that the actual transfer function equals the desired transfer function, where ζ_θ is a design parameter.

(e) Find the DC gain of the inner loop.

(f) Assuming that the inner loop is working properly, replace the inner loop system with its DC gain and draw the block diagram for the outer loop and derive the actual transfer function from ϕ^d to ϕ.

(g) Select the proportional gain k_{p_ϕ} so that θ^d just saturates at A_θ when a step of size $A_\phi = 30$ degrees is placed on ϕ^d.

(h) Find the DC gain of the outer loop. What does it mean that this value does not equal to one?

(i) What are the closed loop poles of the inner and the outer loops?

Homework III.9 Implement the PD and P controllers for the inner and outer loops designed in Homework III.8 on the satellite system in Simulink.

Homework III.10

(a) With PD control for the inner loop, what is the system type of the inner loop? Characterize the steady state error when the reference input is a step, a ramp, and a parabola. How does this change if you add an integrator?

(b) Recall from Problem 6, that in deriving the transfer functions that we obtained the expression

$$\Theta(s) = \left(\frac{1/J_s}{s^2 + b/J_s s + k/J_s} \right) \tau(s) + \left(\frac{b/J_s s + k/J_s}{s^2 + b/J_s s + k/J_s} \right) \Psi(s),$$

where we considered

$$d(s) = \left(\frac{b/J_s s + k/J_s}{s^2 + b/J_s s + k/J_s} \right) \Psi(s)$$

as an output disturbance on the inner loop. What is the response of the inner loop to a step on ψ with and without an integrator in the controller?

(c) With the integrator included, put the characteristic equation of the inner loop in Evan's form and use the Matlab SISO tool to plot the root locus verses the integrator gain k_{i_θ}.

(d) With PD control for the outer loop, what is the system type of the outer loop? Characterize the steady state error when the reference input is a step, a ramp, and a parabola. How does this change if you add an integrator?

(e) With the integrator, put the characteristic equation of the outer loop in Evan's form and use the Matlab SISO tool to plot the root locus verses the integrator gain k_{i_ψ}.

Homework III.11 The objective of this problem is to implement the PID controller using Matlab code using only measured outputs of the system.

(a) Change the parameter file param.m to specify actual parameters and the parameters known to the controller. Change the values for J_s, J_p, k and b known to the controller by 5%, to simulate imperfect knowledge of the plant.

(b) Rearrange the block diagram so that the controller is implemented as an m-function implemented at the sample rate of $T_s = 0.01$. Assume that the controller only has knowledge of the angles ϕ and θ as well as the desired angle ϕ^d.

(c) Implement the nested PID loops designed in Problems III.8 and III.10 using an m-function called satellite_ctrl.m. Use the dirty derivative gain of $\tau = 0.05$.

Homework III.12

(a) Draw by hand the Bode plot of the inner loop transfer function from force F to angle θ for the inverted pendulum. Use the Matlab `bode` command and compare your results.

(b) Draw by hand the Bode plot of the outer loop transfer function from force F to angle θ for the inverted pendulum. Use the Matlab `bode` command and compare your results.

Homework III.13 For the inner loop of the satellite attitude control, use the Matlab `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, (2) the plant under PD control, and (3) the plant under PID control, using the control gains calculated in Homework 11.

(a) To what percent error can the closed loop system under PD control track a step in $\theta^d(t)$?

(b) If the input disturbance has frequency content below 0.02 radians per second, what percentage of the input disturbance appears in the output.

For the outer loop of the satellite attitude control, use the Matlab `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, (2) the plant under P control, and (3) the plant under PI control, using the control gains calculated in Homework 11.

(c) Under PID control, what is the percent tracking error of the outer loop for reference signals $\phi^d(t)$ with frequency content below 0.01 radians per second?

(d) If all of the frequency content of the noise $n(t)$ is greater than $\omega = 1$ radians per second, what percentage of the noise shows up in the output signal θ, using PID control?

Homework III.14 For this homework assignment we will use loopshaping to improve the PD controllers developed in Homework 9. Download the simulink files on the web that are associated with this homework. We will focus on the outer loop. When you open the Simulink file, notice the presence of the LTI system with parameter C.

(a) At the Matlab prompt, type `>> C = 1` to set the additional compensator originally to one. Run the simulation to establish baseline performance.

(b) Note in `param.m` the line `L=series(P_out,C_out_pd)`, where $L(s)$ is the loop gain for the out loop. Load the `sisotool` and enter L for the system plant under the tab `System Data`.

(c) The objective will be to (1) improve disturbance rejection and tracking by a factor of 10 for input signals and disturbance below 0.003 radians/sec, (2) improve noise attention by a factor of 10 for frequencies above 10 radians/sec. In `sisotool` add constraints that reflect these objectives.

(d) Add a PI or phase lag filter to satisfy the disturbance rejection and tracking objective, and add one to two poles to meet the noise rejection objective. Take care to maintain roughly the same cross over frequency and adequate phase margin.

(e) Export $C(s)$, and run the combined controller in Simulink.

Homework III.15 The objective of this problem is to implement state feedback controller using the full state. Start with the simulation files developed in Homework 11. We first note that the there was a mistake in the derivation of the dynamics in the solution to problem 3 where $k(\phi - \theta)$ was not copied to the last line of the solutions. This has propagated through all of the solutions. The correct dynamics should have

been

$$\begin{pmatrix} J_s & 0 \\ 0 & J_p \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \tau - b(\dot{\theta} - \dot{\phi}) - k(\theta - \phi) \\ -b(\dot{\phi} - \dot{\theta}) - k(\phi - \theta) \end{pmatrix}.$$

Therefore, letting $x = (\theta, \phi, \dot{\theta}, \dot{\phi})^{\top}$, $u = \tau$, and $y = (\theta, \phi - \theta)^{\top}$, the state space equations should have been

$$\dot{x} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_s} & -\frac{k}{J_s} & \frac{b}{J_s} & -\frac{b}{J_s} \\ \frac{k}{J_p} & -\frac{k}{J_p} & \frac{b}{J_p} & -\frac{b}{J_p} \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{J_s} \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} x.$$

You will need to modify the `satellite_dynamics.m` to reflect the correct dynamics.

(a) Modify satellite_dynamics.m so that the outputs are $(\theta, \phi, x)^{\top}$, where $x = (\theta, \phi, \dot{\theta}, \dot{\phi})^{\top}$. In this problem we will use x instead of θ and ϕ, but in the next problem we will again remove x as an output of the plant and reconstruct the state using an observer.

(b) Rearrange the block diagram so that the state x is an input to the controller.

(c) Select the closed loop poles as the roots of the equations $s^2 + 2\zeta_\theta \omega_{n_\theta} s + \omega_{n_\theta}^2 = 0$ and $s^2 + 2\zeta_\phi \omega_{n_\phi} s + \omega_{n_\phi}^2 = 0$. Where ω_{n_θ}, ω_{n_θ}, ζ_ϕ, and ζ_ϕ are tuning parameters. Start with the numbers used in problem 11.

(d) Add the state space matrices A, B, C, D to your param file.

(e) Verify that the state space system is controllable by checking that $\text{rank}(\mathcal{C}) = n$.

(f) Using the Matlab `place` command, find the feedback gain K such that the eigenvalues of $(A - BK)$ are equal to desired closed loop poles. Find the reference gain k_r so that the DC-gain from ϕ^d to ϕ is equal to one. In this case, the C matrix used to find k_r is $C = (0, 1, 0, 0)$. Why?

(g) Implement the state feedback scheme in Simulink and tune the closed loop poles to get good response.

Homework III.16

    (a) Modify the state feedback solution developed in Homework 15 to add an integrator with anti-windup to the feedback loop for ϕ.

    (b) Change the parameters known to the controller by $\pm\,5\%$.

    (c) Add a constant input disturbance of 0.5 Newtons-meters to the input of the plant and verify that when the integrator gain is set to zero, there is a steady state tracking error.

    (c) Tune the integrator to get zero steady state tracking performance.

    Hints:

- To get fast integrator action, try turning off the integrator during transition, i.e., when $|\dot{\theta}_1|$ is greater than some limit.

Homework III.17 The objective of this problem is to replace the state in Homework 16 with the estimated state $\hat{x}$ produced by an observer.

    (a) Modify the simulink diagram from Homework 16 so that y is also an input to the controller, as well as x, and so that $\hat{x}$ is also an output of the controller. Add a scope to plot the estimation error $x - \hat{x}$.

    (b) Verify that the state space system is observable by checking that $\text{rank}(\mathcal{O}) = n$.

    (c) In the control block, add an observer to estimate the state $\hat{x}$. Do not yet use $\hat{x}$ in the state feedback equations. Run the controller using the actual state x and tune the observer so that the estimation error is small. Make sure the $\text{eig}(A - BK - LC)$ are in the left half plane.

    (d) Replace x in the controller with the estimated state $\hat{x}$ and tune the controller and observer if necessary to obtain good performance.

    (e) Modify the simulink diagram to add an input disturbance of 0.5 Newton-meters. Add a state to the observer to estimate the input disturbance . Replace the integrator with an estimate of the disturbance and compare the performance.
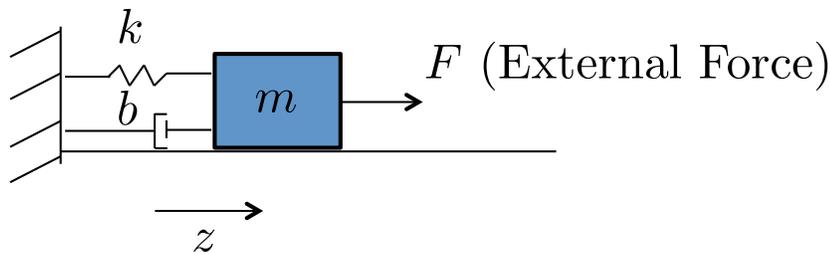
Design Problem 4

Mass Spring Damper

Figure 4.1: Mass Spring Damper

Figure 4.1 shows the mass-spring-damper system. The mass slides along a frictionless level surface and is connected to a wall by a spring with spring constant k and a damper with damping constant b. The position of the mass is given by z, where $z = 0$ is the position where the spring is not stretched. The speed of the mass is given by $\dot{z}$. An external force F is applied to the mass as shown in Figure 4.1.

Assume the following physical constants: $m = 5$ kg, $k = 3$ Kg/s^2, $b = 0.5$ Kg/s.

Homework IV.1 Using the configuration variable z, write an expression for the kinetic energy of the system.

Homework IV.2 Create a simulink animation of the mass-spring-damper system. The input should be a slider for z. Turn in a screen capture of the animation.

Homework IV.3

    (a) Find the potential energy for the system.

    (b) Define the generalized coordinates.

    (c) Find the generalized forces.

    (d) Derive the equations of motion using the Euler-Lagrange equations.

Homework IV.4 Modify the simulink model created in homework IV.2 by creating an s-function that implements the equations of motion. The input to the s-function should be a slider for force. The output should go to the animation developed in homework IV.2.

Homework IV.5

    (a) Find the equilibria of the system.

    (b) Linearize the system about the equilibria.

Homework IV.6

    (a) Using the Laplace transform, convert from time domain to the s-domain.

    (b) Find the transfer function from the small deviation in force $\tilde{F}$ to the small deviation in mass position $\tilde{z}$.

    (c) Draw the associated block diagram.

Homework IV.7 Defining the states as $x = (\tilde{z}, \dot{\tilde{z}})^\top$, the input as $u = \tilde{F}$, and the measured output as $y = \tilde{z}$, find the linear state space equations in the form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

Homework IV.8 Suppose that the size of the input force is limited to $F_{\max} = 2$ N and that the objective is to move the mass in steps from -0.5 meters to 0.5 meters.

(a) Suppose that a PD controller is used to stabilize the system and regulate z to z^d. Draw the (linearized) closed loop block diagram using the transfer function found in problem IV.6. Derive the actual transfer function from z^d to z.

(b) Considering the equilibrium force F_e, derive a bound $F_{\max}$ on the linearized force $\tilde{F}$ such that $\left|\tilde{F}\right| \leq F_{\max}$ when the maximum possible equilibrium Force is used.

(c) Select the proportional gain k_p so that F just saturates when a step of size $A = 1$ meters is placed on z_d, i.e, a step command from $z^d = -0.5$ to $z^d = 0.5$.

(d) If the desired transfer function is given by

$$z(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} z^d(s),$$

find the natural frequency ω_n and the proportional gain k_d so that the actual transfer function equals the desired transfer function, where ζ is a design parameter.

(e) What are the closed loop poles?

Homework IV.9 Implement the PD controller designed in Homework IV.8 on the mass spring damper in Simulink.

Homework IV.10

(a) With PD control, what is the system type? Characterize the steady state error when the reference input is a step, a ramp, and a parabola. How does this change if you add an integrator?

(b) Consider the case where a constant disturbance acts at the input to the plant (for example an inaccurate knowledge of the spring constant in this case). What is the steady state error to a constant input disturbance when the integrator is not present, and when it is present?

(c) Including the integrator, put the characteristic equation in Evan's form and use the Matlab SISO tool to plot the root locus verses the integrator gain k_i.

Homework IV.11 The objective of this problem is to implement the PID
controller using Matlab code using only measured outputs of the sys-
tem.

  (a) Change the parameter file param.m to specify actual parameters
       and the parameters known to the controller. Change the values of
       m, k, and b known to the controller by 5% to simulate imperfect
       knowledge of the plant.

  (b) Rearrange the block diagram so that the controller is implemented
       as an m-function implemented at the sample rate of $T_s = 0.01$.
       The controller should only assume knowledge of the position z and
       the desired position z^d.

  (c) Implement the PID controller designed in Problems IV.8 and IV.10
       using an m-function called mass_ctrl.m. Use the dirty derivative
       gain of $\tau = 0.05$. Tune the integrators to so that there is no steady
       state error.

Homework IV.12 Draw by hand the Bode plot of the mass spring damper
from force $\tilde{F}$ to position $\tilde{z}$. Use the Matlab `bode` command and com-
pare your results.

Homework IV.13 For the mass spring, use the Matlab `bode` command to
create a graph that simultaneously displays the Bode plots for (1) the
plant, (2) the plant under PD control, and (3) the plant under PID
control, using the control gains calculated in Homework 11.

  (a) What is the tracking error to a unit step under PD control?

  (b) What is the tracking error to a unit ramp under PID control? -be
       careful-

  (c) If the frequency content of the input disturbance $d_i(t)$ is below $\omega = 0.1$ radians per second, what percentage of the input disturbance
       shows up in the output z under PD control?

  (d) If all of the frequency content of the noise $n(t)$ is greater than
       $\omega = 100$ radians per second, what percentage of the noise shows
       up in the output signal z?

Homework IV.14 For this homework assignment, we will use the loopshaping design methodology to improve the PID controllers developed earlier in this class. Given time constraints we have not talked about digital implementation or anti-windup schemes for loopshaping. Therefore, the controllers will be implemented using a transfer function. Download the simulink files on the web that are associated with this homework. When you open the Simulink file, notice the presence of the LTI system with input C.

(a) At the Matlab prompt, type >> C = 1 to set the additional compensator originally to one. Run the simulation to establish baseline performance.

(b) Note in `param.m` the line `L=series(C_pid,G)`, where $L(s)$ is the loop gain. Load the `sisotool` and enter L for the system plant under the tab `System Data`.

(c) The objective will be to (1) improve disturbance rejection and tracking so that the tracking error to a step is about 3% of the input, (2) improve noise attention by a factor of 10 for frequencies above 200 radians/sec. In `sisotool` add constraints that reflect these objectives.

(d) Add a phase lag filter to satisfy the disturbance rejection and tracking objective, and add poles to meet the noise rejection objective. Maintain a phase margin around 60 degrees.

(e) Export $C(s)$, and run the combined controller in Simulink.

Turn the following graphics: (1) The step response with PID only. (3) The loopshape in `sisotool` showing constraints and the loop gain that satisfies the contraints. (4) The final step response.

Homework IV.15 The objective of this problem is to implement state feedback controller using the full state. Start with the simulation files developed in Homework Problem 11.

(a) Modify `mass_dynamics.m` so that the outputs are $(z, x)^\top$, where $x = (z, \dot{z})^\top$. In this problem we will use x instead of z but in the next problem we will again remove x as an output of the plant and reconstruct the state using an observer.

(b) Rearrange the block diagram so that the state x is an input to the controller.

(c) Select the closed loop poles as the roots of the equations $s^2 + 2\zeta\omega_n + \omega_n^2 = 0$ where ω_n, and ζ were found in Homework Problem 11.

(d) Add the state space matrices A, B, C, D derived in Homework 7 to your param file.

(e) Verify that the state space system is controllable by checking that $\text{rank}(\mathcal{C}) = n$.

(f) Using the Matlab `place` command, find the feedback gain K such that the eigenvalues of $(A - BK)$ are equal to desired closed loop poles. Find the reference gain k_r so that the DC-gain from z^c to z is equal to one. Note that $K = (k_p, k_d)$ where k_p and k_d are the proportional and derivative gains found in Homework 11. Why?

(g) Implement the state feedback scheme in Simulink and tune the closed loop poles to get good response.

Hints:

- For comparison, using exact parameters, I get the following state space model:

$$A = \begin{pmatrix} 0 & 1.0000 \\ -0.6000 & -0.1000 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0.2000 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

- As for the equilibrium force, convince yourself that the following are equivalent:

$$F_e = kz^d$$
$$x_e = (z^d, 0)^\top$$
$$F = F_e - K(x - x_e)$$

or

$$F = k_r z^d - Kx.$$

Homework IV.16

- **(a)** Modify the state feedback solution developed in Homework 15 to add an integrator with anti-windup to the feedback loop for z.

- **(b)** Change the parameters known to the controller by $\pm\,5\%$ and verify that there is a steady state tracking error when the integrator gain is zero.

- **(c)** Tune the integrator to get zero steady state tracking performance.

Hints:

- To get fast integrator action, try turning off the integrator during transition, i.e., when $|\dot{z}|$ is greater than some limit.

Homework IV.17 The objective of this problem is to replace the state x in Homework 16 with the estimated state $\hat{x}$ produced by an observer.

- **(a)** Modify the simulink diagram from Homework 16 so that y is also an input to the controller, as well as x, and so that $\hat{x}$ is also an output of the controller. Add a scope to plot the estimation error $x - \hat{x}$.

- **(b)** Verify that the state space system is observable by checking that $\mathrm{rank}(\mathcal{O}) = n$.

- **(c)** In the control block, add an observer to estimate the state $\hat{x}$. Do not yet use $\hat{x}$ in the state feedback equations. Run the controller using the actual state x and tune the observer so that the estimation error is small. Make sure the $\mathrm{eig}(A - BK - LC)$ are in the left half plane.

- **(d)** Replace x in the controller with the estimated state $\hat{x}$ and tune the controller and observer if necessary to obtain good performance.

- **(e)** Add a state to the observer to estimate the disturbance. Replace the integrator with an estimate of the disturbance and compare the performance.

Hints:

- *Note that the observer is estimating the linearized state $\tilde{x} = x - x_e$. Therefore the observer update equation will be*

$$\dot{\hat{\tilde{x}}} = A\hat{\tilde{x}} + B\tilde{u} + L(\tilde{y} - C\hat{\tilde{x}}),$$

  but because $\dot{\hat{\tilde{x}}} = \dot{x} - \dot{x}_e = \dot{x}$, and since $y_e = Cx_e$, this is equivalent to

$$\dot{\hat{x}} = A(\hat{x} - x_e) + B(u - u_e) + L(y - C\hat{x}).$$

- *For this problem, where the primary input disturbance is due to the parameter mismatch, the integrator seems to work better than the disturbance estimator.*
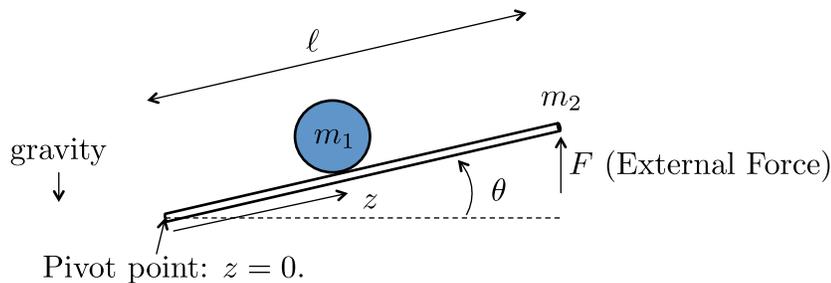
Design Problem 5

Ball on Beam

Figure 5.1: Ball on Beam Problem

Figure 5.1 shows the ball and beam system. The position of the ball measured from the pivot point is z and the speed of the ball along the direction of the beam is $\dot{z}$. The angle of the beam from level is θ and the angular speed of the beam is $\dot{\theta}$. Gravity acts in the down direction. The mass of the ball is m_1 and the mass of the beam is m_2. The length of the beam is ℓ. An external force is applied at the end of the beam as shown in Figure 5.1.

Use the following physical parameters: $m_1 = 0.35$ kg, $m_2 = 2$ kg, $\ell = 0.5$ m, $g = 9.8$ m/s^2.

Homework V.1 Using the configuration variable z and θ, write an expression for the kinetic energy of the system.

Homework V.2 Create a simulink animation of the ball on beam. The inputs should be sliders for z and θ. Turn in a screen capture of the animation.

Homework V.3

    (a) Find the potential energy for the system.

    (b) Define the generalized coordinates.

    (c) Find the generalized forces.

    (d) Derive the equations of motion using the Euler-Lagrange equations.

Homework V.4 Modify the simulink model created in homework V.2 by creating an s-function that implements the equations of motion. The input to the s-function should be a slider for force. The output should go to the animation developed in homework V.2.

Homework V.5

    (a) Find the equilibria of the system.

    (b) Linearize the system about the equilibria.

Homework V.6

    (a) Using the Laplace transform, convert from time domain to the s-domain.

    (b) Find the transfer functions that define the cascade connected from force F to the beam angle $\tilde{\theta}$ to the ball position $\tilde{z}$.

    (c) Find explicit expressions for the disturbances that have been ignored.

    (d) Draw the associated block diagram.

Homework V.7 Defining the states as $x = (\tilde{z}, \tilde{\theta}, \dot{\tilde{z}}, \dot{\tilde{\theta}})^{\top}$, the input as $u = F$, and the measured output as $y = (\tilde{z}, \tilde{\theta})^{\top}$, find the linear state space equations in the form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

Homework V.8 Suppose that the size of the input force on the cart is limited to $F_{\max} = 15$ N. Design separate PD controllers for the inner loop and the outer loop.

(a) Using the transfer functions derived in problem V.6, and using a PD controller for the inner loop to regulate $\tilde{\theta}$ to $\tilde{\theta}^d$, and a PD controller for the outer loop to regulate z to z^d, draw the block diagram of the complete closed loop system.

(b) Derive the actual transfer function of the inner loop from $\tilde{\theta}^d$ to $\tilde{\theta}$.

(c) Considering the equilibrium force F_e, derive a bound $F_{\max}$ on the linearized force $\tilde{F}$ such that $\left|\tilde{F}\right| \leq F_{\max}$ when the maximum possible equilibrium force is used.

(d) Select the proportional gain k_{p_θ} so that $\tilde{F}$ just saturates when a step of size A_{th} is placed on $\tilde{\theta}_d$.

(e) If the desired transfer function is given by

$$\tilde{\theta}(s) = \frac{\omega_{n_\theta}^2}{s^2 + 2\zeta_\theta \omega_{n_\theta} s + \omega_{n_\theta}^2} \tilde{\theta}^d(s),$$

find the natural frequency ω_{n_θ} and the derivative gain k_{d_θ} so that the actual transfer function equals the desired transfer function, where ζ_θ is a design parameter.

(f) What are the closed loop poles of the inner loop?

(g) Assuming that the inner loop is working properly and that $\tilde{\theta} \approx \tilde{\theta}^d$, draw the block diagram for the outer loop and derive the actual transfer function from $\tilde{z}^d$ to $\tilde{z}$.

(h) Select the proportional gain k_{p_z} so that $\tilde{\theta}^d$ just saturates at A_{th} when a step of size $A_z = 0.25$ meters is placed on $\tilde{z}^d$.

(i) If the desired transfer function is given by

$$\tilde{z}(s) = \frac{\omega_{n_z}^2}{s^2 + 2\zeta_z \omega_{n_z} s + \omega_{n_z}^2} \tilde{z}^d(s),$$

find the natural frequency ω_{n_z} and the derivative gain k_{d_z} so that the actual transfer function equals the desired transfer function, where ζ_z is a design parameter.

(j) What are the closed loop poles of the outer loop?

Homework V.9 Implement the PD controllers for the inner and outer loops designed in Homework V.8 on the ball and beam in Simulink.

Homework V.10

(a) With PD control for the inner loop, what is the system type of the inner loop? Characterize the steady state error when the reference input is a step, a ramp, and a parabola.

(b) Recall from Problem 6, that in deriving the transfer functions that we obtained the expression

$$\Theta(s) = \frac{a_1}{s^2} F(s) - \frac{a_2}{s^2} \tilde{Z}(s)$$

where we considered

$$d(s) = -\frac{a_2}{s^2} \tilde{Z}(s)$$

as an output disturbance on the inner loop. What is the response of the inner loop to a step on $\tilde{z}$ with and without an integrator in the controller?

(c) With the integrator included, put the characteristic equation of the inner loop in Evan's form and use the Matlab SISO tool to plot the root locus verses the integrator gain k_{i_θ}.

(d) With PD control for the outer loop, what is the system type of the outer loop? Characterize the steady state error when the reference input is a step, a ramp, and a parabola.

Homework V.11 The objective of this problem is to implement the PID controller using Matlab code using only measured outputs of the system.

(a) Change the parameter file param.m to specify actual parameters and the parameters known to the controller. Change the values for m_1, m_2, and ℓ known to the controller by 5% to simulate imperfect knowledge of the plant.

(b) Rearrange the block diagram so that the controller is implemented as an m-function implemented at the sample rate of $T_s = 0.01$. Implement the nested PID loops in Problems V.8 and V.10 using

an m-function called ballbeam_ctrl.m. Use the dirty derivative gain of $\tau = 0.05$. Tune the integrator to so that there is no steady state error. The controller should only assume knowledge of the position z, the angle θ, and the desired position z^d.

(c) Note that the integrator gain will need to be negative which will cause problems for the anti-windup scheme that we have been implementing. Remove the old anti-windup scheme and implement a new scheme where the integrator only winds up when $|\dot{z}|$ is small.

Homework V.12

(a) Draw by hand the Bode plot of the inner loop transfer function from force F to angle $\tilde{\theta}$ for the ball beam system. Use the Matlab `bode` command and compare your results.

(b) Draw by hand the Bode plot of the outer loop transfer function from angle $\tilde{\theta}$ to position $z(t)$ for the ball beam system. Use the Matlab `bode` command and compare your results.

Homework V.13 For the inner loop of the ballbeam system, use the Matlab `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, and (2) the plant under PD control, using the control gains calculated in Homework 11.

(a) To what percent error can the closed loop system under PD control track the desired input if all of the frequency content of $\theta^d(t)$ is below $\omega = 1.0$ radians per second?

(b) If the frequency content of the input disturbance is all contained below 0.8 radians per second, what percentage of the input disturbance shows up in the output?

(c) If all of the frequency content of the noise $n(t)$ is greater than $\omega = 300$ radians per second, what percentage of the noise shows up in the output signal θ?

For the outer loop of the inverted pendulum, use the Matlab `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, (2) the plant under PD control, and and (3) the plant under PID control, using the control gains calculated in Homework 11.

(d) If the frequency content of an output disturbance is contained below 0.1 radian/sec, what percentage of the output disturbance will be contained in the output under both PD and PID control?

(e) If the reference signal is $y^d(t) = 2\sin(0.2t)$ what is the output error under PD control?

Homework V.14 For this homework assignment we will use loopshaping to improve the PD controllers developed in Homework V.9. Download the simulink files on the web that are associated with this homework. We will focus entirely on the outer loop. When you open the Simulink file, notice the presence of the LTI system with input C.

(a) At the Matlab prompt, type `>> C = 1` to set the additional compensator originally to one. Run the simulation to establish baseline performance.

(b) Note in `param.m` the line `L=series(P_out,C_out_pd)`, where $L(s)$ is the loop gain for the out loop. Load the `sisotool` and enter L for the system plant under the tab `System Data`.

(c) The objective will be to (1) improve disturbance rejection and tracking by a factor of 10 for input signals and disturbance below 0.01 radians/sec, (2) improve noise attention by a factor of 10 for frequencies above 200 radians/sec. In `sisotool` add constraints that reflect these objectives.

(d) Add a phase lag filter to satisfy the disturbance rejection and tracking objective, and add one to two poles to meet the noise rejection objective. Maintain a phase margin around 60 degrees.

(e) Export $C(s)$, and run the combined controller in Simulink.

Turn the following graphics: (1) The step response with PID only. (3) The loopshape in `sisotool` showing constraints and the loop gain that satisfies the contraints. (4) The final step response.

Homework V.15 The objective of this problem is to implement state feedback controller using the full state. Start with the simulation files developed in Homework Problem 11.

(a) Modify ballbeam_dynamics.m so that the outputs are $(\theta, z, x)^\top$, where $x = (\theta, z, \dot\theta, \dot z)^\top$. In this problem we will use x instead of

z and θ, but in the next problem we will again remove x as an output of the plant and reconstruct the state using an observer.

(b) Rearrange the block diagram so that the state x is an input to the controller.

(c) Using the values for ω_{n_z}, ζ_z, ω_{n_θ}, and ζ_θ found in problem 11 as starting points, find the desired closed loop poles.

(d) Add the state space matrices A, B, C, D derived in Homework 7 to your param file.

(e) Verify that the state space system is controllable by checking that $\mathrm{rank}(\mathcal{C}) = n$.

(f) Using the Matlab `place` command, find the feedback gain K such that the eigenvalues of $(A - BK)$ are equal to desired closed loop poles. Find the reference gain k_r so that the DC-gain from z^c to z is equal to one.

(g) Implement the state feedback scheme in Simulink and tune the closed loop poles to get good response.

Hints:

- Assume that the parameters known to the controller equal the true parameters. Otherwise there will be a steady state error. In the next homework we will add an integrator to remove the steady state error.

- Make sure that the states are in the right order.

- You will need to use the equilibrium Force as a feedforward term but there are two ways that you can do it. Convince yourself that the following are equivalent:

$$F_e = \frac{m_1 g z_d}{L} + \frac{1}{2} m_2 g$$
$$x_e = (0, z^c, 0, 0)^\top$$
$$F = F_e - K(x - x_e)$$

or

$$F_e = \frac{1}{2} m_2 g$$
$$F = F_e + k_r z^c - Kx.$$

- There are several mistakes in previous solutions. For comparison, using exact parameters, I get the following state space model:

$$A = \begin{pmatrix} 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0 & -18.1923 & 0 & 0 \\ -9.8 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 2.6519 \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

- Be careful in computing $k_r = -1/C(A-BK)^{-1}B$. In this formula, C is the output matrix matching the reference input. Therefore, for this problem $C = (0, 1, 0, 0)$.

Homework V.16

(a) Modify the state feedback solution developed in Homework 15 to add an integrator with anti-windup to the feedback loop for z.

(b) Change the parameters known to the controller by $\pm 5\%$ and verify that there is a steady state tracking error when the integrator gain is zero.

(c) Tune the integrator to get zero steady state tracking performance.

Hints:

- To get fast integrator action, try turning off the integrator during transition, i.e., when $|\dot{z}|$ is greater than some limit.

Homework V.17 RWB: Skip this problem. Do III.17 instead. The objective of this problem is to replace the state in Homework 16 with the estimated state $\hat{x}$ produced by an observer.

(a) Modify the simulink diagram from Homework 16 so that y is also an input to the controller, as well as x, and so that $\hat{x}$ is also an

output of the controller. Add a scope to plot the estimation error $x - \hat{x}$.

(b) Verify that the state space system is observable by checking that rank($\mathcal{O}$) = n.

(c) In the control block, add an observer to estimate the state $\hat{x}$. Do not yet use $\hat{x}$ in the state feedback equations. Run the controller using the actual state x and tune the observer so that the estimation error is small. Make sure the eig($A - BK - LC$) are in the left half plane.

(d) Replace x in the controller with the estimated state $\hat{x}$ and tune the controller and observer if necessary to obtain good performance.

(e) Modify the simulink diagram to add an input disturbance of 0.2 Newton-meters. Add a state to the observer to estimate the input disturbance . Replace the integrator with an estimate of the disturbance and compare the performance.

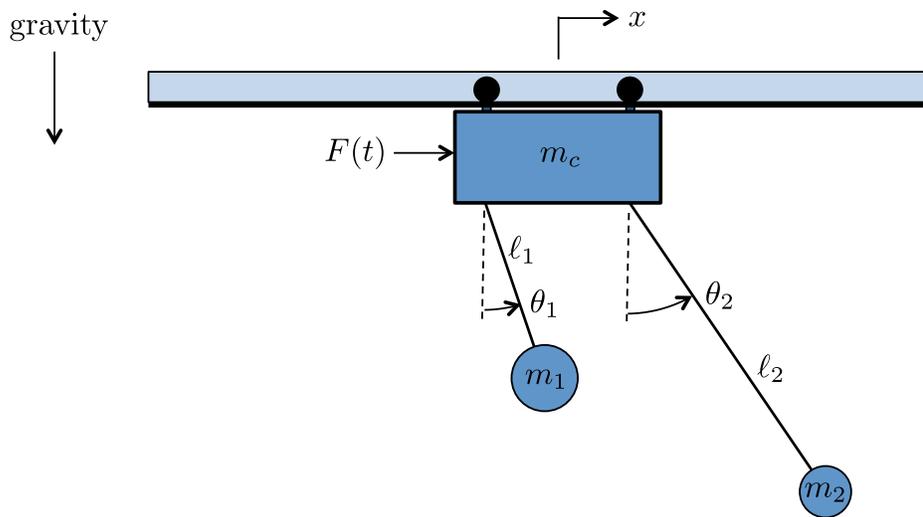# Design Problem 6

Case Study VI. Gantry Crane

Figure 6.1: Gantry Crane

Figure 6.1 shows a gantry crane system with two slung loads. The crane car is suspended from a track and has mass m_c. The rolling friction on the track is proportional the the velocity of car with friction constant b. The crane carries two loads. Load 1 has mass m_1 and is suspended by a cable of length ℓ_1. Load 2 has mass m_2 and is suspended by a cable of length ℓ_2. The crane car is propelled along the track by a force $F(t)$ and its position is given by x. The angles of the loads are given by θ_1 and θ_2 respectively.

Use the following physical parameters: $m_c = 1500$ kg, $m_1 = 500$ kg,

$m_2 = 200$ kg, $b = 200$ N-s/m, $\ell_1 = 10$ m, $\ell_2 = 24$ m, $g = 9.81$ m/s^2.

Homework VI.1 Using the configuration variable x, θ_1, and θ_2, write an expression for the kinetic energy of the system.

Homework VI.2 Create a simulink animation of the gantry crane system. The inputs should be sliders for x, θ_1 and θ_2. Turn in a screen capture of the animation.

Homework VI.3

(a) Find the potential energy for the system.

(b) Define the generalized coordinates.

(c) Find the generalized forces.

(d) Derive the equations of motion using the Euler-Lagrange equations.

Homework VI.4 Modify the simulink model created in homework VI.2 by creating an s-function that implements the equations of motion. The input to the s-function should be a slider for F. The output should go to the animation developed in homework VI.2.

Homework VI.5

(a) Find the equilibria of the system.

(b) Linearize the system about the equilibria.

Homework VI.6

(a) Using the formula $A^{-1} = \frac{\text{adj}(A)}{\det(A)}$ where $\text{adj}(A)$ is the adjugate of A which equals transpose of the co-factors of A, and $\det(A)$ is the determinant of A, show that

$$M^{-1} = \begin{pmatrix} \frac{1}{m_c} & -\frac{1}{m_c\ell_1} & -\frac{1}{m_c\ell_2} \\ -\frac{1}{m_c\ell_1} & \frac{m_c+m_1}{m_cm_1\ell_1^2} & \frac{1}{m_c\ell_1\ell_2} \\ -\frac{1}{m_c\ell_2} & \frac{1}{m_c\ell_1\ell_2} & \frac{m_c+m_2}{m_cm_2\ell_2^2} \end{pmatrix}$$

(b) Using M^{-1} solve for $(\ddot{x}, \ddot{\theta}_1, \ddot{\theta}_2)^{\top}$.

(c) Using the Laplace transform, convert the equations of motion from time domain to the s-domain.

(d) Solve each equation individually for $X(s)$, $\Theta_1(s)$, and $\Theta_2(s)$ ignoring the coupling between equations.

(e) For frequency domain designs, we will treat the motion of the first mass m_1, or in other words θ_1 as a disturbance on the system. Identify the transfer function from force $F(s)$ and the second angle $\Theta_2(s)$ and the transfer function from $\Theta_2(s)$ to the cart position $X(s)$. Label the other terms as disturbances.

(f) Draw a block diagram of the open-loop system showing the transfer function cascade with F as the input and x as the output of the cascade, with disturbances entering at the appropriate locations.

Homework VI.7 Defining the states as $x = (x, \theta_1, \theta_2, \dot{x}, \dot{\theta}_1, \dot{\theta}_2)^\top$, the input as $u = F$, and the measured output as $y = (x, \theta_1, \theta_2)^\top$, find the linear state space equations in the form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

Homework VI.8 Suppose that the size of the input force on the cart is limited to $F_{\max} = 4500$ N. Design separate PD controllers for the inner loop and the outer loop.

(a) Using the transfer functions derived in problem VI.6, and using a PD controller for the inner loop to regulate θ_2 to θ_2^d, and a PD controller for the outer loop to regulate z to z^d, draw the block diagram of the complete closed loop system.

(b) Derive the actual transfer function of the inner loop from θ_2^d to θ_2.

(c) Select the proportional gain $k_{p_{\theta_2}}$ so that F just saturates when a step of size A_{th} is placed on θ_2^d.

(d) If the desired transfer function is given by

$$\tilde{\theta}(s) = \frac{\omega_{n\theta}^2}{s^2 + 2\zeta_\theta \omega_{n\theta} s + \omega_{n\theta}^2} \tilde{\theta}^d(s),$$

find the natural frequency $\omega_{n_{\theta_2}}$ and the derivative gain $k_{d_{\theta_2}}$ so that the actual poles of the closed loop transfer function equals

the poles of the desired transfer function, where ζ_{θ_2} is a design parameter. Note that the DC gain of the actual transfer function is not equal to one.

(e) Compute the DC gain k_{DC}.

(f) Assuming that the inner loop is much faster than the outer loop, replace the inner loop with its DC gain k_{DC}, and draw the block diagram for the outer loop. Derive the actual transfer function from z^d to z. For this part of the problem you will need to use the transfer function $T_{z/\theta_2} = \frac{-\frac{m_2 g}{m_c}}{s(s+\frac{b}{m_c})}$ instead of $T_{z/\theta_2} = \frac{\frac{m_2 g}{m_c}}{s(s+\frac{b}{m_c})}$. The negative sign makes sense physically since achieving a positive motion in z requires a negative motion in θ_2, but at this time I am not completely sure why the negative sign does not show up in the model.

(g) Select the proportional gain k_{p_z} so that θ_2^d just saturates at A_{th} when a step of size $A_z = 0.25$ meters is placed on z^d.

(h) If the desired transfer function is given by

$$z(s) = \frac{\omega_{n_z}^2}{s^2 + 2\zeta_z \omega_{n_z} s + \omega_{n_z}^2} z^d(s),$$

find the natural frequency ω_{n_z} and the derivative gain k_{d_z} so that the actual transfer function equals the desired transfer function, where ζ_z is a design parameter.

(i) What are the closed loop poles of the outer loop?

Homework VI.9 Implement the PD controllers for the angle and position loops designed in Homework VI.8 on the gantry crane system in Simulink. Note that the derivative of the positions (θ_2 and x) are available for feedback as outputs of your dynamics block.

Homework VI.10

(a) With PD control for the inner loop, what is the system type of the inner loop? Characterize the steady state error when the reference input is a step, a ramp, and a parabola. How does this change if you add an integrator?

(b) Recall from Problem 6, that in deriving the transfer functions that we obtained the expression (setting $\Theta_1(s) = 0$)

$$\Theta_2(s) = \left(\frac{-a_1}{s^2 + a_2}\right) F(s) + \left(\frac{a_4 s}{s^2 + a_2}\right) Z(s),$$

where we considered

$$d(s) = \left(\frac{a_4 s}{s^2 + a_2}\right) Z(s)$$

as an output disturbance on the inner loop. What is the response of the inner loop to a step on $z(t)$ with and without an integrator in the controller?

(c) With PD control for the outer loop, what is the system type of the outer loop? Characterize the steady state error when the reference input is a step, a ramp, and a parabola. How does this change if you add an integrator?

(d) Find the response of the outer loop to a unit step input disturbance, both when the outer loop controller does not have an integrator and when it does.

(e) With the integrator, put the characteristic equation of the outer loop in Evan's form and use the Matlab SISO tool to plot the root locus verses the integrator gain k_i.

Homework VI.11 The objective of this problem is to implement the PID controller using Matlab code using only measured outputs of the system.

(a) Change the parameter file param.m to specify actual parameters and the parameters known to the controller. Change the values for m_1, m_2, m_c, ℓ_1, ℓ_2, and b known to the controller by $\pm 5\%$ to simulate imperfect knowledge of the plant.

(b) Rearrange the block diagram so that the controller is implemented as an m-function implemented at the sample rate of $T_s = 0.01$. The controller should only assume knowledge of the position z and the angles θ_1, and θ_2, as well as the desired position z^d.

(c) Implement the nested PID loops designed in Problems VI.8 and VI.10 using an m-function called gantry_ctrl.m. Use the dirty derivative gain of $\tau = 0.05$.

Homework VI.12

(a) Draw by hand the Bode plot of the inner loop transfer function from force F to angle $\tilde{\theta}_2$ for the ball beam system. Use the Matlab `bode` command and compare your results.

(b) Draw by hand the Bode plot of the outer loop transfer function from angle $\tilde{\theta}$ to position $z(t)$ for the ball beam system. Use the Matlab `bode` command and compare your results.

Homework VI.13 For the inner loop of the gantry system, use the Matlab `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, (2) the plant under PD control, and and (3) the plant under PID control using the control gains calculated in Homework 11.

(a) What percent of the input disturbance shows up in $\theta_2(t)$ if the frequency content of the input disturbance is below 0.3 radians per second?

(b) What is the tracking error for a unit step in the command to $\theta_2(t)$ for PD control? How does this relate to the DC gain of the system?

(c) If all of the frequency content of the noise $n(t)$ is greater than $\omega = 100$ radians per second, what percentage of the noise shows up in the output signal θ_2?

For the outer loop of the inverted pendulum, use the Matlab `bode` command to create a graph that simultaneously displays the Bode plots for (1) the plant, and (2) the plant under PD control, using the control gains calculated in Homework 11.

(d) If the frequency content of an output disturbance is contained below 0.01 radian/sec, what percentage of the output disturbance will be contained in the output under PD control?

(e) If the reference signal $z^d(t)$ is a ramp with slope 0.1 m/s, what is the steady state tracking error under PD control?

Homework VI.14 For this homework assignment we will use loopshaping to improve the PD controllers developed in Homework V.9. Download the simulink files on the web that are associated with this homework. We will focus on the outer loop. When you open the Simulink file, notice the presence of the LTI system with parameter C.

(a) At the Matlab prompt, type `>> C = 1` to set the additional compensator originally to one. Run the simulation to establish baseline performance.

(b) Note in `param.m` the line `L=series(P_out,C_out_pd)`, where $L(s)$ is the loop gain for the out loop. Load the `sisotool` and enter L for the system plant under the tab `System Data`.

(c) The objective will be to (1) improve disturbance rejection and tracking by a factor of 10 for input signals and disturbance below 0.001 radians/sec, (2) improve noise attention by a factor of 10 for frequencies above 20 radians/sec. In `sisotool` add constraints that reflect these objectives.

(d) Add a phase lag filter to satisfy the disturbance rejection and tracking objective, and add one to two poles to meet the noise rejection objective. The phase margin in this case is interpreted different. The phase above 0 degrees corresponds to the normal phase margin. This should remain around 60 degrees. To understand why, you will need to understand the Nyquist criteria.

(e) Export $C(s)$, and run the combined controller in Simulink.

Turn the following graphics: (1) The step response with PID only. (3) The loopshape in `sisotool` showing constraints and the loop gain that satisfies the contraints. (4) The final step response.

Homework VI.15 The objective of this problem is to implement state feedback controller using the full state. Start with the simulation files developed in Homework Problem 11.

(a) Modify gantry_dynamics.m so that the outputs are $(z, \theta_1, \theta_2, x)^\top$, where $x = (z, \theta_1, \theta_2, \dot{z}, \dot{\theta}_1, \dot{\theta}_2)^\top$. In this problem we will use the full state x, but in the next problem we will again remove x as an output of the plant and reconstruct the state using an observer.

(b) Rearrange the block diagram so that there is a single controller with the desired position z^d and the state x is an input to the controller.

(c) There will be six closed loop poles, and in this problem we can select all six poles, or in other words, we will be controlling both angles. Using the values for ω_{n_z}, ζ_z, $\omega_{n_{\theta_2}}$, and ζ_{θ_2} derived in Problem 11 as starting points, find desired closed loop poles for all six states.

(d) Add the state space matrices A, B, C, D derived in Homework 7 to your param file. For comparison, for perfect parameters, I get

$$
A = \begin{pmatrix}
0 & 0 & 0 & 1.0000 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.0000 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.0000 \\
0 & 3.2700 & 1.3080 & -0.1333 & 0 & 0 \\
0 & -13.0800 & -1.3080 & 0.1333 & 0 & 0 \\
0 & -0.1363 & -0.4632 & 0.0056 & 0 & 0
\end{pmatrix}
$$

$$
B = \begin{pmatrix}
0 \\
0 \\
0 \\
0.6667e - 03 \\
-0.6667e - 03 \\
-0.0278e - 03
\end{pmatrix}
$$

$$
C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.
$$

(e) Verify that the state space system is controllable by checking that $\text{rank}(\mathcal{C}) = n$.

(f) Using the Matlab `place` command, find the feedback gain K such that the eigenvalues of $(A - BK)$ are equal to desired closed loop poles. Use the control signal $u = -Kx - k_r z^d$ where z^d is the desired position of the cart.

(g) Implement the state feedback scheme in Simulink and tune the closed loop poles to get good response.

Homework VI.16

(a) Modify the state feedback solution developed in Homework 15 to add an integrator with anti-windup to the position feedback.

(b) Change the parameters known to the controller by $\pm$ 5%.

(c) Add a constant input disturbance of 1000 Newtons to the input of the plant and verify that when the integrator gain is set to zero, there is a steady state tracking error.

(c) Tune the integrator to get zero steady state tracking performance.

Hints:

- The integrator gain will be negative.

- To get fast integrator action, try turning off the integrator during transition, i.e., when $|\dot{y}|$ is greater than some limit.

Homework VI.17 The objective of this problem is to replace the state in Homework 16 with the estimated state $\hat{x}$ produced by an observer.

(a) Modify the simulink diagram from Homework 16 so that y is also an input to the controller, as well as x, and so that $\hat{x}$ is also an output of the controller, where $y = (z, \theta_1, \theta_2)^\top$. Add a scope to plot the estimation error $x - \hat{x}$.

(b) Verify that the state space system is observable by checking that rank$(\mathcal{O}) = n$.

(c) In the control block, add an observer to estimate the state $\hat{x}$. Do not yet use $\hat{x}$ in the state feedback equations. Run the controller using the actual state x and tune the observer so that the estimation error is small. Make sure the eig$(A - BK - LC)$ are in the left half plane.

(d) Replace x in the controller with the estimated state $\hat{x}$ and tune the controller and observer if necessary to obtain good performance.

(e) Modify the simulink diagram to add an input disturbance of 1000 Newtons. Add a state to the observer to estimate the input disturbance . Replace the integrator with an estimate of the disturbance and compare the performance.

Hints:

- *The performance of the system is highly dependent on the pole locations of the observer. I obtained good response by using the LQR command*

  `P.Lo = lqr(P.Ao', P.Co', diag([1,1,1,1,1,1,rho]),diag([1,1,1]))'`,

  where `rho` *encodes the weight on the estimation error of the disturbance and needs to be a large number.*