CHAPTER TWO

RIGID MOTIONS AND HOMOGENEOUS TRANSFORMATIONS

A large part of robot kinematics is concerned with the establishment of various coordinate systems to represent the positions and orientations of rigid objects and with transformations among these coordinate systems. Indeed, the geometry of three-dimensional space and of rigid motions plays a central role in all aspects of robotic manipulation. In this chapter we study the operations of rotation and translation and introduce the notion of homogeneous transformations.[1] Homogeneous transformations combine the operations of rotation and translation into a single matrix multiplication, and are used in Chapter Three to derive the so-called forward kinematic equations of rigid manipulators. We also investigate the transformation of velocities and accelerations among coordinate systems. These latter quantities are used in subsequent chapters to study the velocity kinematics in Chapter Five, including the derivation of the manipulator Jacobian, and also to derive the dynamic equations of motion of rigid manipulators in Chapter Six.

[1] Since we make extensive use of elementary matrix theory, the reader may wish to review Appendix A before beginning this chapter.

2.1 ROTATIONS

Figure 2-1 shows a rigid object S to which a coordinate frame $ox_1y_1z_1$ is attached. We wish to relate the coordinates of a point $\mathbf{p}$ on S in the $ox_1y_1z_1$ frame to the coordinates of $\mathbf{p}$ in a fixed (or nonrotated) reference frame $ox_0y_0z_0$. Let $\{\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0\}$ denote the standard orthonormal basis in $ox_0y_0z_0$; thus $\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0$ are unit vectors along the x_0, y_0, z_0 axes, respectively. Similarly, let $\{\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1\}$ be the standard orthonormal basis in $ox_1y_1z_1$. Then the vector from the common origin to the point $\mathbf{p}$ on the object can be represented either with respect to $ox_0y_0z_0$ as

$$\mathbf{p}_0 = p_{0x}\mathbf{i}_0 + p_{0y}\mathbf{j}_0 + p_{0z}\mathbf{k}_0 \tag{2.1.1}$$

or with respect to $ox_1y_1z_1$ as

$$\mathbf{p}_1 = p_{1x}\mathbf{i}_1 + p_{1y}\mathbf{j}_1 + p_{1z}\mathbf{k}_1 \tag{2.1.2}$$

Since $\mathbf{p}_0$ and $\mathbf{p}_1$ are representations of the same vector $\mathbf{p}$, the relationship between the components of $\mathbf{p}$ in the two coordinate frames can be obtained as follows.

$$\begin{aligned} p_{0x} &= \mathbf{p}_0 \cdot \mathbf{i}_0 = \mathbf{p}_1 \cdot \mathbf{i}_0 \\ &= p_{1x}\mathbf{i}_1 \cdot \mathbf{i}_0 + p_{1y}\mathbf{j}_1 \cdot \mathbf{i}_0 + p_{1z}\mathbf{k}_1 \cdot \mathbf{i}_0 \end{aligned} \tag{2.1.3}$$

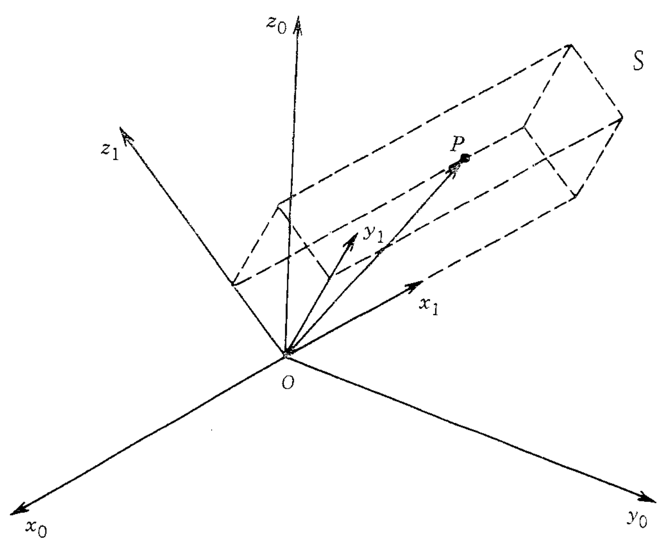

FIGURE 2-1
Coordinates frame attached to a rigid body

We have similar formulas for p_{0y} and p_{0z}, namely

$$p_{0y} = p_{1x}\mathbf{i}_1\cdot\mathbf{j}_0 + p_{1y}\mathbf{j}_1\cdot\mathbf{j}_0 + p_{1z}\mathbf{k}_1\cdot\mathbf{j}_0 \quad (2.1.4)$$

$$p_{0z} = p_{1x}\mathbf{i}_1\cdot\mathbf{k}_0 + p_{1z}\mathbf{j}_1\cdot\mathbf{k}_0 + p_{1z}\mathbf{k}_1\cdot\mathbf{k}_0 \quad (2.1.5)$$

We may write the above three equations together as

$$\mathbf{p}_0 = R_0^1\mathbf{p}_1 \quad (2.1.6)$$

where

$$R_0^1 = \begin{bmatrix} \mathbf{i}_1\cdot\mathbf{i}_0 & \mathbf{j}_1\cdot\mathbf{i}_0 & \mathbf{k}_1\cdot\mathbf{i}_0 \\ \mathbf{i}_1\cdot\mathbf{j}_0 & \mathbf{j}_1\cdot\mathbf{j}_0 & \mathbf{k}_1\cdot\mathbf{j}_0 \\ \mathbf{i}_1\cdot\mathbf{k}_0 & \mathbf{j}_1\cdot\mathbf{k}_0 & \mathbf{k}_1\cdot\mathbf{k}_0 \end{bmatrix} \quad (2.1.7)$$

The 3×3 matrix represents the transformation matrix from the coordinates of $\mathbf{p}$ with respect to the frame $ox_1y_1z_1$ to the coordinates with respect to the frame $ox_0y_0z_0$. Thus, if a given point is expressed in $ox_1y_1z_1$-coordinates as $\mathbf{p}_1$ then $R_0^1\mathbf{p}_1$ represents the **same vector** expressed relative to the $ox_0y_0z_0$-coordinate frame.

Similarly we can write

$$p_{1x} = \mathbf{p}_1\cdot\mathbf{i}_1 = \mathbf{p}_0\cdot\mathbf{i}_1 \quad (2.1.8)$$

$$= p_{0x}\mathbf{i}_0\cdot\mathbf{i}_1 + p_{0y}\mathbf{j}_0\cdot\mathbf{i}_1 + p_{0z}\mathbf{k}_0\cdot\mathbf{i}_1$$

etc., or in matrix form

$$\mathbf{p}_1 = R_1^0\mathbf{p}_0 \quad (2.1.9)$$

where

$$R_1^0 = \begin{bmatrix} \mathbf{i}_0\cdot\mathbf{i}_1 & \mathbf{j}_0\cdot\mathbf{i}_1 & \mathbf{k}_0\cdot\mathbf{i}_1 \\ \mathbf{i}_0\cdot\mathbf{j}_1 & \mathbf{j}_0\cdot\mathbf{j}_1 & \mathbf{k}_0\cdot\mathbf{j}_1 \\ \mathbf{i}_0\cdot\mathbf{k}_1 & \mathbf{j}_0\cdot\mathbf{k}_1 & \mathbf{k}_0\cdot\mathbf{k}_1 \end{bmatrix} \quad (2.1.10)$$

Thus the matrix R_1^0 represents the inverse of the transformation R_0^1. Since the inner product is commutative, i.e., $\mathbf{i}_0\cdot\mathbf{j}_0 = \mathbf{j}_0\cdot\mathbf{i}_0$, etc., we see that

$$R_1^0 = (R_0^1)^{-1} = (R_0^1)^T \quad (2.1.11)$$

Such a matrix R_0^1 whose inverse is its transpose is said to be **orthogonal**. The column vectors of R_0^1 are of unit length and mutually orthogonal (Problem 2-1). It can also be shown (Problem 2-2) that $\det R_0^1 = \pm1$. If we restrict ourselves to right-handed coordinate systems, as defined in Appendix A, then $\det R_0^1 = +1$ (Problem 2-3). For simplicity we refer to orthogonal matrices with determinant +1 as **rotation matrices**. It is customary to refer to the set of **all** 3×3 rotation matrices by the symbol $SO(3)$.[2]

[2]The notation $SO(3)$ stands for Special Orthogonal group of order 3.

(i) Example 2.1.1

Suppose the frame $ox_1y_1z_1$ is rotated through an angle θ about the z_0 axis, and it is desired to find the resulting transformation matrix R_0^1. Note that by convention the positive sense for the angle θ is given by the right hand rule; that is, a positive rotation of θ degrees about the z-axis would advance a right-hand threaded screw along the positive z-axis. From Figure 2-2 we see that

$$\mathbf{i}_0 \cdot \mathbf{i}_1 = \cos\theta \qquad \mathbf{j}_1 \cdot \mathbf{i}_0 = -\sin\theta \tag{2.1.12}$$

$$\mathbf{j}_0 \cdot \mathbf{j}_1 = \cos\theta \qquad \mathbf{i}_1 \cdot \mathbf{j}_0 = \sin\theta$$

$$\mathbf{k}_0 \cdot \mathbf{k}_1 = 1$$

and all other dot products are zero. Thus the transformation R_0^1 has a particularly simple form in this case, namely

$$R_0^1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.1.13}$$

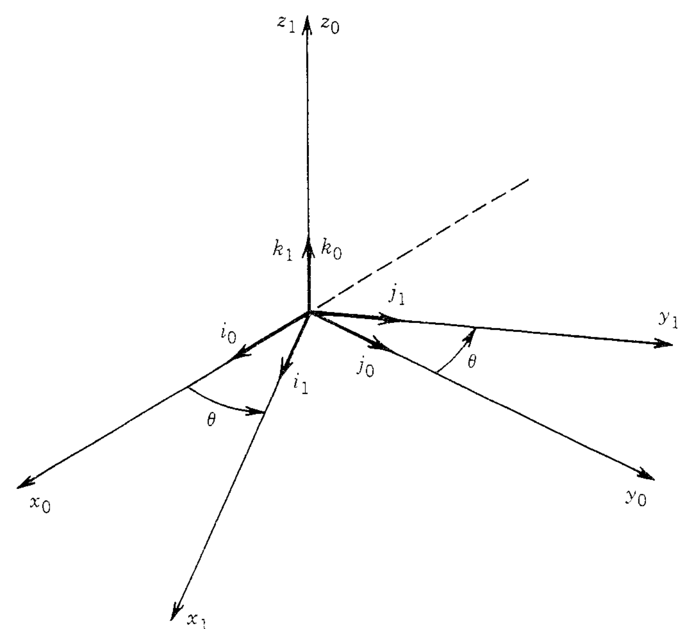

FIGURE 2-2
Rotation about the z_0 axis.

The transformation (2.1.13) is called a **basic rotation matrix** (about the z-axis). In this case we find it useful to use the more descriptive notation $R_{z,\theta}$ instead of R_0^1 to denote the matrix (2.1.13). It is easy to verify that the basic rotation matrix $R_{z,\theta}$ has the properties

$$R_{z,0} = I \tag{2.1.14}$$

$$R_{z,\theta}R_{z,\phi} = R_{z,\theta+\phi} \tag{2.1.15}$$

which together imply

$$R_{z,\theta}^{-1} = R_{z,-\theta} \tag{2.1.16}$$

Similarly the basic rotation matrices representing rotations about the x and y axes are given as (Problem 2-5)

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{2.1.17}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{2.1.18}$$

which also satisfy properties analogous to (2.1.14)–(2.1.16).

We may also interpret a given rotation matrix as specifying the *orientation* of the coordinate frame $ox_1y_1z_1$ relative to the frame $ox_0y_0z_0$. In fact, the columns of R_0^1 are the direction cosines of the coordinate axes in $ox_1y_1z_1$ relative to the coordinate axes of $ox_0y_0z_0$. For example, the first column $(\mathbf{i}_1\cdot\mathbf{i}_0, \mathbf{i}_1\cdot\mathbf{j}_0, \mathbf{i}_1\cdot\mathbf{k}_0)^T$ of R_0^1 specifies the direction of the x_1-axis relative to the $ox_0y_0z_0$ frame.

(ii) Example 2.1.2

Consider the frames $ox_0y_0z_0$ and $ox_1y_1z_1$ shown in Figure 2-3. Projecting the unit vectors $\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1$ onto $\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0$ gives the coordinates of $\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1$ in the $ox_0y_0z_0$ frame. We see that the coordinates of $\mathbf{i}_1$ are $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})^T$, the coordinates of $\mathbf{j}_1$ are $(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})^T$ and the coordinates of $\mathbf{k}_1$ are $(0, 1, 0)^T$. The rotation matrix R_0^1 specifying the orientation of $ox_1y_1z_1$ relative to $ox_0y_0z_0$ has these as its column vectors, that is,

$$R_0^1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix} \tag{2.1.19}$$

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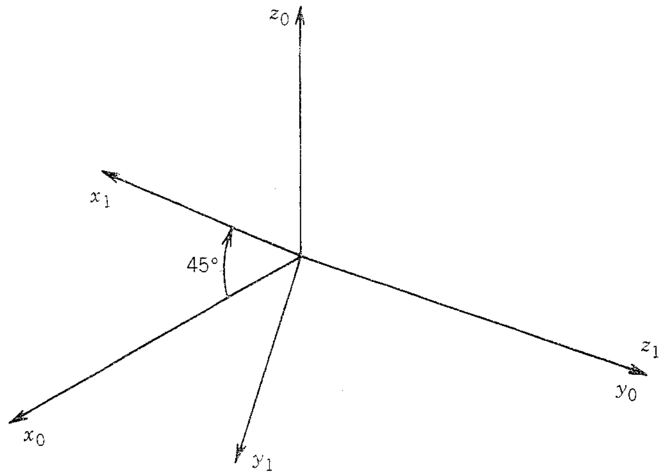

FIGURE 2-3
Defining the relative orientation of two frames.

A third interpretation of a rotation matrix $R \in SO(3)$ is as an operator acting on vectors in a fixed frame $ox_0y_0z_0$. In other words, instead of relating the coordinates of a fixed vector with respect to two different coordinate frames, the expression (2.1.10) can represent the coordinates in $ox_0y_0z_0$ of a point $\mathbf{p}_1$ which is obtained from a point $\mathbf{p}_0$ by a given rotation.

(*iii*) ***Example 2.1.3***

The vector $\mathbf{p}_0 = (1, 1, 0)^T$ is rotated about the y_0-axis by $\frac{\pi}{2}$ as shown in Figure 2-4. The resulting vector $\mathbf{p}_1$ is given by

$$\mathbf{p}_1 = R_{y,\frac{\pi}{2}}\mathbf{p}_0 \tag{2.1.20}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

2.1.1 SUMMARY

We have seen that a rotation matrix $R \in SO(3)$ can be interpreted in three distinct ways:

1. It represents a coordinate transformation relating the coordinates of a point $\mathbf{p}$ in two different frames.

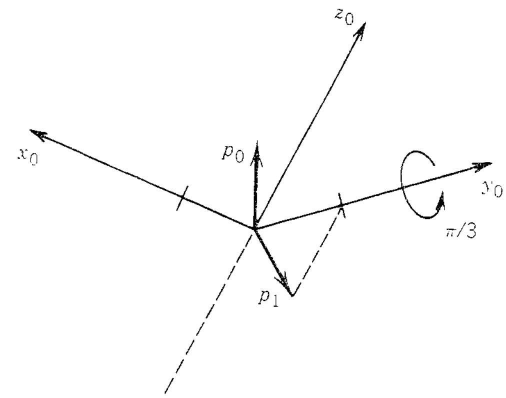

FIGURE 2-4
Rotating a vector about an axis.

2. It gives the orientation of a transformed coordinate frame with respect to a fixed coordinate frame.
3. It is an operator taking a vector $\mathbf{p}$ and rotating it to a new vector $R\mathbf{p}$ in the same coordinate system.

The particular interpretation of a given rotation matrix R that is being used must then be made clear by the context.

2.2 COMPOSITION OF ROTATIONS

In this section we discuss the composition of rotations. It is important for subsequent chapters that the reader understand the material in this section thoroughly before moving on. Recall that the matrix R_0^1 in equation (2.1.6) represents a rotational transformation between the frames $ox_0y_0z_0$ and $ox_1y_1z_1$. Suppose we now add a third coordinate frame $ox_2y_2z_2$ related to the frames $ox_0y_0z_0$ and $ox_1y_1z_1$ by rotational transformations. A given point $\mathbf{p}$ can then be represented in three ways: $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$ in the three frames. The relationship between these representations of $\mathbf{p}$ is

$$\mathbf{p}_0 = R_0^1\mathbf{p}_1 \tag{2.2.1}$$

$$\mathbf{p}_0 = R_0^2\mathbf{p}_2 \tag{2.2.2}$$

$$\mathbf{p}_1 = R_1^2\mathbf{p}_2 \tag{2.2.3}$$

where each R_i^j is a rotation matrix. Note that R_0^1 and R_0^2 represent rotations relative to the $ox_0y_0z_0$ axes, while R_1^2 represents a rotation relative to the $ox_1y_1z_1$ frame. Substituting (2.2.3) into (2.2.1) yields

$$\mathbf{p}_0 = R_0^1 R_1^2\mathbf{p}_2 \tag{2.2.4}$$

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Comparing (2.2.2) and (2.2.4) we have the identity

$$R_0^2 = R_0^1 R_1^2 \tag{2.2.5}$$

Equation 2.2.5 is the composition law for rotational transformations. It states that, in order to transform the coordinates of a point $\mathbf{p}$ from its representation $\mathbf{p}_2$ in the $ox_2y_2z_2$-frame to its representation $\mathbf{p}_0$ in the $ox_0y_0z_0$-frame, we may first transform to its coordinates $\mathbf{p}_1$ in the $ox_1y_1z_1$-frame using R_1^2 and then transform $\mathbf{p}_1$ to $\mathbf{p}_0$ using R_0^1.

We may interpret Equation 2.2.4 as follows. Suppose initially that all three of the coordinate frames coincide. We first rotate the frame $ox_1y_1z_1$ relative to $ox_0y_0z_0$ according to the transformation R_0^1. Then, with the frames $ox_1y_1z_1$ and $ox_2y_2z_2$ coincident, we rotate $ox_2y_2z_2$ relative to $ox_1y_1z_1$ according to the transformation R_1^2. In each case we call the frame relative to which the rotation occurs the **current frame**.

(i) *Example 2.2.1*

Henceforth, whenever convenient we use the shorthand notation $c_\theta = \cos\theta$, $s_\theta = \sin\theta$ for trigonometric functions. Suppose a rotation matrix R represents a rotation of ϕ degrees about the current y-axis followed by a rotation of θ degrees about the current z axis. Then the matrix R is given by

$$R = R_{y,\phi} R_{z,\theta} \tag{2.2.6}$$

$$= \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_\phi c_\theta & -c_\phi s_\theta & s_\phi \\ s_\theta & c_\theta & 0 \\ -s_\phi c_\theta & s_\phi s_\theta & c_\phi \end{bmatrix}$$

It is important to remember that the order in which a sequence of rotations are carried out, and consequently the order in which the rotation matrices are multiplied together, is crucial. The reason is that rotation, unlike position, is not a vector quantity and is therefore **not** subject to the laws of vector addition, and so rotational transformations do not commute in general.

(ii) *Example 2.2.2*

Suppose that the above rotations are performed in the reverse order, that is, first a rotation about the current z-axis followed by a rotation about the current y-axis.

Then the resulting rotation matrix is given by

$$R' = R_{z,\theta}R_{y,\phi} \tag{2.2.7}$$

$$= \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix}$$

$$= \begin{bmatrix} c_\phi c_\theta & -s_\theta & s_\phi c_\theta \\ s_\theta c_\phi & c_\theta & s_\theta s_\phi \\ -s_\phi & 0 & c_\phi \end{bmatrix}$$

Comparing (2.2.6) and (2.2.7) we see that $R \neq R'$.

Many times it is desired to perform a sequence of rotations, each about a given fixed coordinate frame, rather than about successive current frames. For example we may wish to perform a rotation about the x_0 axis followed by a rotation about the y_0 (and not y_1!) axis. We will refer to $ox_0y_0z_0$ as the **fixed frame**. In this case the above composition law is not valid. It turns out that the correct composition law in this case is simply to multiply the successive rotation matrices **in the reverse order** from that given by (2.2.5). Note that the rotations themselves are not performed in reverse order. Rather they are performed about the fixed frame rather than about the current frame.

(*iii*) *Example 2.2.3*

Suppose that a rotation matrix R represents a rotation of ϕ degrees about the y_0-axis followed by a rotation of θ about the fixed z_0-axis. Refer to Figure 2-5. Let $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$ be representations of a vector $\mathbf{p}$ as shown.

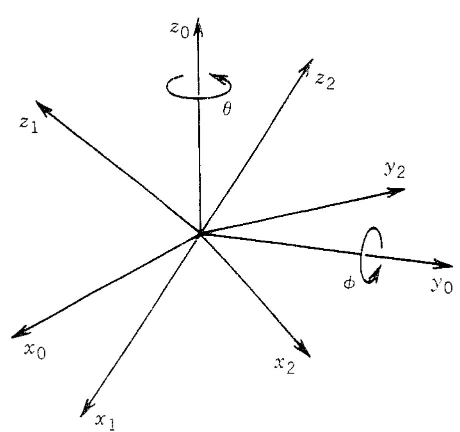

FIGURE 2-5
Composition of rotations.

Initially the fixed and current axes are the same, namely $ox_0y_0z_0$, and therefore we can write as before

$$\mathbf{p}_0 = R_{y,\phi}\mathbf{p}_1 \tag{2.2.8}$$

where $R_{y,\phi}$ is the basic rotation matrix about the y-axis. Now, since the second rotation is about the fixed frame $ox_0y_0z_0$ and not the current frame $ox_1y_1z_1$, we **cannot conclude** that

$$\mathbf{p}_1 = R_{z,\theta}\mathbf{p}_2 \tag{2.2.9}$$

since this would require that we interpret $R_{z,\theta}$ as being a rotation about z_1. In order to use our previous composition law we need somehow to have the fixed and current frames, in this case z_0 and z_1, coincident. Therefore we need first to **undo** the previous rotation, then rotate about z_0 and finally reinstate the original transformation, that is,

$$\mathbf{p}_1 = R_{y,-\phi}R_{z,\theta}R_{y,\phi}\mathbf{p}_2 \tag{2.2.10}$$

This is the correct expression, and not (2.2.9). Now, substituting (2.2.10) into (2.2.8) we obtain

$$\begin{aligned}\mathbf{p}_0 &= R_{y,\phi}\mathbf{p}_1 \qquad (2.2.11)\\ &= R_{y,\phi}R_{y,-\phi}R_{z,\theta}R_{y,\phi}\mathbf{p}_2\\ &= R_{z,\theta}R_{y,\phi}\mathbf{p}_2\end{aligned}$$

It is not necessary to remember the above derivation, only to note by comparing (2.2.11) with (2.2.6) that we obtain the same basic rotation matrices in the reverse order.

We can be summarize the rule of composition of rotational transformations by the following recipe.

Given a fixed frame $ox_0y_0z_0$, a current frame $ox_1y_1z_1$, together with rotation matrix R_0^1 relating them, if a third frame $ox_2y_2z_2$ is obtained by a rotation R_1^2 performed relative to the **current frame** then **postmultiply** R_0^1 by R_1^2 to obtain

$$R_0^2 = R_0^1 R_1^2 \tag{2.2.12}$$

If the second rotation is to be performed relative to the **fixed frame** then **premultiply** R_0^1 by R_1^2 to obtain

$$R_0^2 = R_1^2 R_0^1 \tag{2.2.13}$$

In each case R_0^2 represents the transformation between the frames $ox_0y_0z_0$ and $ox_2y_2z_2$. The frame $ox_2y_2z_2$ that results in (2.2.12) will be different from that resulting from (2.2.13).

2.2.1 ROTATION ABOUT AN ARBITRARY AXIS

Rotations are not always performed about the principal coordinate axes. We are often interested in a rotation about an arbitrary axis in space. Therefore let $\mathbf{k} = (k_x, k_y, k_z)^T$, expressed in the frame $ox_0y_0z_0$, be a unit vector defining an axis. We wish to derive the rotation matrix $R_{\mathbf{k},\theta}$ representing a rotation of θ degrees about this axis.

There are several ways in which the matrix $R_{\mathbf{k},\theta}$ can be derived. Perhaps the simplest way is to rotate the vector **k** into one of the coordinate axes, say z_0, then rotate about z_0 by θ and finally rotate **k** back to its original position. Referring to Figure 2-6 we see that we can rotate **k** into z_0 by first rotating about z_0 by $-\alpha$, then rotating about y_0 by $-\beta$. Since all rotations are performed relative to the fixed frame $ox_0y_0z_0$ the matrix $R_{\mathbf{k},\theta}$ is obtained as

$$R_{\mathbf{k},\theta} = R_{z,\alpha}R_{y,\beta}R_{z,\theta}R_{y,-\beta}R_{z,-\alpha} \tag{2.2.14}$$

From Figure 2-6, since **k** is a unit vector, we see that

$$\sin\alpha = \frac{k_y}{\sqrt{k_x^2+k_y^2}} \tag{2.2.15}$$

$$\cos\alpha = \frac{k_x}{\sqrt{k_x^2+k_y^2}}$$

$$\sin\beta = \sqrt{k_x^2+k_y^2}$$

$$\cos\beta = k_z$$

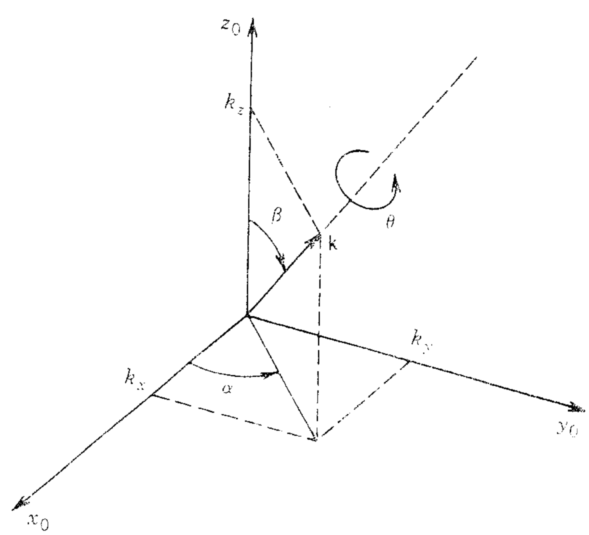

FIGURE 2-6
Rotation about an arbitrary axis.

Substituting (2.2.15) into (2.2.14) we obtain after some lengthy calculation (Problem 2-9)

$$R_{\mathbf{k},\theta} = \begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix} \qquad (2.2.16)$$

where $v_\theta = \text{vers}\,\theta = 1 - c_\theta$.

2.3 FURTHER PROPERTIES OF ROTATIONS

The nine elements r_{ij} in a general rotational transformation R as in (2.1.7) are not independent quantities. Indeed a rigid body possess at most three rotational degrees-of-freedom and thus at most three quantities are required to specify its orientation. In this section we derive three ways in which an arbitrary rotation can be represented using only three independent quantities. The first is the **axis/angle** representation. The second is the **Euler Angle** representation and the third is the **roll-pitch-yaw** representation.

2.3.1 AXIS/ANGLE REPRESENTATION

A rotation matrix $R \in SO(3)$ can always be represented by a single rotation about a suitable axis in space by a suitable angle as

$$R = R_{\mathbf{k},\theta} \qquad (2.3.1)$$

where $\mathbf{k}$ is a unit vector defining the axis of rotation, and θ is the angle of rotation about $\mathbf{k}$. Equation (2.3.1) is called the **axis-angle representation** of R. Given an arbitrary rotation matrix R with components (r_{ij}), the equivalent angle θ and equivalent axis $\mathbf{k}$ are given by the expressions [2]

$$\theta = \cos^{-1}\left(\frac{Tr(R) - 1}{2}\right) \qquad (2.3.2)$$

$$= \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

where Tr denotes the trace of R, and

$$\mathbf{k} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \qquad (2.3.3)$$

The axis/angle representation is not unique since a rotation of $-\theta$ about $-\mathbf{k}$ is the same as a rotation of θ about $\mathbf{k}$, that is,

$$R_{\mathbf{k},\theta} = R_{-\mathbf{k},-\theta} \qquad (2.3.4)$$

If $\theta = 0$ then R is the identity matrix and the axis of rotation is undefined.

(i) *Example 2.3.1*

Suppose R is generated by a rotation of 90^o about z_0 followed by a rotation of 30^o about y_0 followed by a rotation of 60^o about x_0. Then

$$R = R_{x,60}R_{y,30}R_{z,90} \tag{2.3.5}$$

$$= \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ \frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix}$$

We see that $Tr(R) = 0$ and hence the equivalent angle is given by (2.3.2) as

$$\theta = \cos^{-1}(-\frac{1}{2}) = 120^o \tag{2.3.6}$$

The equivalent axis is given from (2.3.3) as

$$\mathbf{k} = (\frac{1}{\sqrt{3}}\ ,\ \frac{1}{2\sqrt{3}} - \frac{1}{2}\ ,\ \frac{1}{2\sqrt{3}} + \frac{1}{2})^T \tag{2.3.7}$$

The above axis/angle representation characterizes a given rotation by four quantities, namely the three components of the equivalent axis $\mathbf{k}$ and the equivalent angle θ. However, since the equivalent axis $\mathbf{k}$ is given as a unit vector only two of its components are independent. The third is constrained by the condition that $\mathbf{k}$ is of unit length, Therefore, only three independent quantities are required in this representation of a rotation R. We can represent the equivalent angle/axis by a single vector $\mathbf{r}$ as

$$\mathbf{r} = (r_x\ ,\ r_y\ ,\ r_z)^T = (\theta k_x\ ,\ \theta k_y\ ,\ \theta k_z)^T \tag{2.3.8}$$

Note, since $\mathbf{k}$ is a unit vector, that the length of the vector $\mathbf{r}$ is the equivalent angle θ and the direction of $\mathbf{r}$ is the equivalent axis $\mathbf{k}$.

2.3.2 EULER ANGLES

A more common method of specifying a rotation matrix in terms of three independent quantities is to use the so-called Euler Angles. Consider again the fixed coordinate frame $ox_0y_0z_0$ and the rotated frame $ox_1y_1z_1$ shown in Figure 2-7.

We can specify the orientation of the frame $ox_1y_1z_1$ relative to the frame $ox_0y_0z_0$ by three angles (θ, ϕ, ψ), known as Euler Angles, and obtained by three successive rotations as follows: First rotate about the z axis by the angle ϕ. Next rotate about the current y axis by the angle θ.

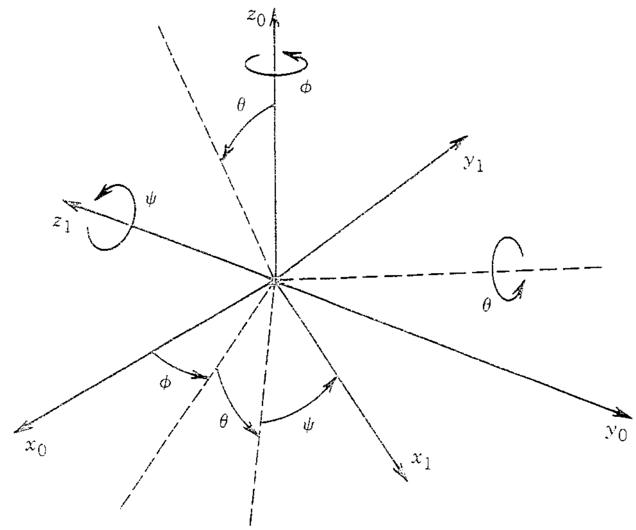

FIGURE 2-7
Euler angle representation.

Finally rotate about the current z by the angle ψ. In terms of the basic rotation matrices (2.1.14)–(2.1.16) the resulting rotational transformation R_0^1 can be generated as the product

$$R_0^1 = R_{z,\phi} R_{y,\theta} R_{z,\psi} \tag{2.3.9}$$

$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta s_\psi & s_\theta c_\psi & c_\theta \end{bmatrix}$$

In Chapter Four we study the inverse problem of finding the Euler Angles (θ,ϕ,ψ) given an arbitrary rotation matrix R.

2.3.3 ROLL, PITCH, YAW ANGLES

A rotation matrix R can also be described as a product of successive rotations about the principal coordinate axes x_0, y_0, and z_0 taken in a specific order. These rotations define the **roll**, **pitch**, and **yaw** angles, which we shall also denote ϕ,θ,ψ, and which are shown in Figure 2-8.

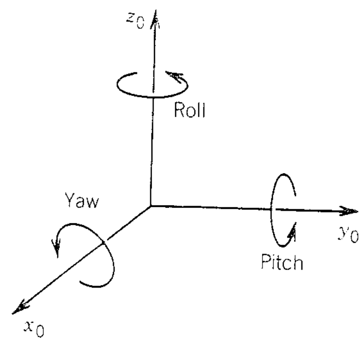

FIGURE 2-8
Roll, pitch and yaw angles.

We specify the order of rotation as $x-y-z$, in other words, first a yaw about the x_0-axis through an angle ψ, then pitch about the y_0-axis an angle θ, and finally roll about the z_0-axis an angle ϕ. Since the successive rotations are relative to the fixed frame, the resulting transformation matrix is given by

$$R_0^1 = R_{z,\phi}R_{y,\theta}R_{x,\psi} \tag{2.3.10}$$

$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}$$

$$= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

Of course, instead of yaw-pitch-roll relative to the fixed frames we could also interpret the above transformation as roll-pitch-yaw, in that order, each taken with respect to the current frame. The end result is the same matrix (2.3.10).

2.4 HOMOGENEOUS TRANSFORMATIONS

Consider now a coordinate system $o_1x_1y_1z_1$ obtained from $o_0x_0y_0z_0$ by a parallel translation of distance $|\mathbf{d}|$ as shown in Figure 2-9. Thus $\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0$ are parallel to $\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1$, respectively. The vector $\mathbf{d}_0^1$ is the vector from the origin o_0 to the origin o_1 expressed in the coordinate system $o_0x_0y_0z_0$.