ECEn 671 | SOLUTIONS | HOMEWORK #1

2.1-6:

IN ORDER TO DEMONSTRATE THAT $d_m(x,y)$ IS A METRIC ON $\mathbb{X}$, WE NEED TO SHOW IT SATISFIES EACH OF THE PROPERTIES OF A METRIC.

1. $d(x,y) = d(y,x)$

SINCE d IS A METRIC

$$d_m(y,x) = \min(1, d(y,x)) = \min(1, d(x,y)) = d_m(x,y) \checkmark$$

2. $d(x,y) \geq 0$

SINCE $d(x,y) \geq 0$

$$d_m(y,x) = \min(1, d(x,y)) \geq \min(1, 0) = 0 \checkmark$$

3. $d(x,y) = 0$ IFF $x = y$

SUPPOSE $d_m(x,y) = 0$. THEN $d_m(x,y) = 0 = \min(1, d(x,y))$.

$\min(1, d(x,y)) = 0$ IFF $d(x,y) = 0$, AND $d(x,y) = 0$ IFF $x = y$, SO

$d_m(x,y) = 0$ IFF $x = y$. $\checkmark$

4. FOR ALL $x, y, z \in \mathbb{X}$, $d(x,z) \leq d(x,y) + d(y,z)$

SO WE NEED TO SHOW:

$$d_m(x,z) \leq d_m(x,y) + d_m(y,z)$$

OR:

$$\min(1, d(x,z)) \leq \min(1, d(x,y)) + \min(1, d(y,z))$$

LET'S LOOK AT THE DIFFERENT CASES: (8 TOTAL)

$\Rightarrow$ IF $d(x,z) \geq 1$, $d(x,y) \geq 1$, AND $d(y,z) \geq 1$

THEN WE HAVE:

$1 \leq 1 + 1$ WHICH IS OBVIOUSLY TRUE.

$\Rightarrow$ IF $d(x,z) < 1$, $d(x,y) < 1$, AND $d(y,z) < 1$

WE HAVE:

$d(x,z) \leq d(x,y) + d(y,z)$ WHICH IS TRUE SINCE d IS A METRIC AND SATISFIES THE TRIANGLE INEQUALITY.

$\Rightarrow$ IF $d(x,z) < 1$, $d(x,y) \geq 1$ AND $d(y,z) \geq 1$

WE HAVE:

$d(x,z) \leq 1 + 1$ WHICH IS TRUE SINCE $d(x,z) < 1$.

$\Rightarrow$ IF $d(x,z) < 1$, $d(x,y) \geq 1$, AND $d(y,z) < 1$

WE HAVE:

$d(x,z) \leq 1 + d(y,z)$ WHICH IS TRUE SINCE $d(x,z) < 1$ AND $d(y,z) > 0$.

$\Rightarrow$ SAME ARGUMENT FOR $d(x,z)<1$, $d(x,y)<1$, AND $d(y,z) \geq 1$.

$\Rightarrow$ IF $d(x,y) \geq 1$, $d(x,y) < 1$, AND $d(y,z) < 1$

WE HAVE:

$$1 \leq d(x,y) + d(y,z)$$

BUT WE KNOW

$$d(x,y) \leq d(x,y) + d(y,z) \quad \text{AND} \quad 1 \leq d(x,y)$$

SO:

$$1 \leq d(x,y) \leq d(x,y) + d(y,z) \checkmark$$

$\Rightarrow$ IF $d(x,y) \geq 1$, $d(y,y) \geq 1$, $d(x,y) < 1$

WE HAVE:

$$1 \leq 1 + d(x,y)$$ WHICH IS TRUE SINCE $d(x,y) \geq 0$.

$\Rightarrow$ SAME ARGUMENT FOR $d(x,y) \geq 1$, $d(x,y) < 1$, $d(x,y) \geq 1$

THAT COVERS ALL OF THE CASES AND THE TRIANGLE INEQUALITY HELD IN EACH CASE.

SO d_m SATISFIES ALL OF THE PROPERTIES OF A METRIC, AND IS THUS A METRIC.

THE "SIGNIFICANT FEATURE" IS THAT THE METRIC IS ALWAYS BOUNDED BETWEEN 0 AND 1.

2.1-7:

SEQUENCE $\underline{x} = \{x_1, x_2, \ldots, x_n\}$ WHERE $x_n = \dfrac{1}{n+1}$

SEQUENCE $\underline{y} = \{y_1, y_2, \ldots, y_n\}$ WHERE $y_n = \dfrac{n}{n+1}$

$$d_\infty(x,y) \equiv \sup_n |x_n - y_n| = \sup_n \left| \frac{1}{n+1} - \frac{n}{n+1} \right| = \sup_n \left| \frac{1-n}{n+1} \right|$$

$$= \sup_n \left(\frac{n-1}{n+1} \right) \text{ FOR } n \geq 1 \quad = 1 \text{ FOR } n \geq 1$$

SO:

$$d_\infty(x,y) = 1$$

REMEMBER, SUP IS A "LEAST UPPER BOUND", WHICH IS 1 FOR $\frac{n-1}{n+1}$, $n \geq 1$.

BUT:

$$|x_n - y_n| = \left| \frac{1}{n+1} - \frac{n}{n+1} \right| = \left| \frac{1-n}{n+1} \right| = \frac{n-1}{n+1} \quad \text{FOR } n \geq 1$$

SO NOW WE CAN SEE THAT:

$$d_\infty(x,y) = 1 < |x_n - y_n| = \frac{n-1}{n+1} \quad \text{FOR } n \geq 1.$$

SINCE THIS CAN NEVER REACH 1 FOR ANY $n \geq 1$.

2.2-27:

A SET T IS LINEARLY INDEPENDENT IF, FOR EACH $\underline{x} \in T$, $\underline{x}$ IS NOT A LINEAR COMBINATION OF THE POINTS IN THE SET $T - \{\underline{x}\}$.

WE NEED TO PROVE THAT DEFINITION 2.17 IMPLIES THIS DEFINITION, WHICH SUFFICES TO SHOW THAT THE DEFINITIONS ARE EQUIVALENT.

PROOF:

FIRST, SUPPOSE T IS LINEARLY INDEPENDENT, AND LET $\underline{x} \in T$.

NOW LET'S SUPPOSE THAT $\underline{x}$ IS A LINEAR COMBINATION OF POINTS IN $T - \{\underline{x}\}$. THEN FOR SOME POINTS $\underline{x}_1, \ldots, \underline{x}_m \in T$, WE HAVE:

$$\underline{x} = a_1 \underline{x}_1 + a_2 \underline{x}_2 + \ldots + a_m \underline{x}_m$$

OR:

$$a_1 \underline{x}_1 + a_2 \underline{x}_2 + \ldots + a_m \underline{x}_m - \underline{x} = 0$$

BUT THIS COEFFICIENT IS NON-ZERO, VIOLATING THE HYPOTHESIS THAT T IS LINEARLY INDEPENDENT!

SO WE'VE SHOWN SO FAR THAT:

> T LINEARLY INDEPENDENT IMPLIES $\underline{x}$ IS NOT A LINEAR COMBINATION OF POINTS IN $T - \{\underline{x}\}$.

CONVERSELY, SUPPOSE $\underline{x}$ IS NOT A LINEAR COMBINATION OF POINTS IN $T - \{\underline{x}\}$ FOR ALL $\underline{x} \in T$.

NOW LET'S SUPPOSE THAT T IS LINEARLY DEPENDENT. THEN THERE ARE A SET OF VECTORS $\underline{x}_1, \ldots, \underline{x}_m \in T$ AND A SET OF NON-ZERO SCALARS SUCH THAT:

$$a_1 \underline{x}_1 + a_2 \underline{x}_2 + \ldots + a_m \underline{x}_m = 0$$

SUPPOSE WE TAKE $\underline{x} = \underline{x}_1$ (SINCE $\underline{x}$ CAN BE ANY ELEMENT IN T). THEN WE CAN WRITE:

$$\underline{x} = \underline{x}_1 = \frac{a_2}{a_1} \underline{x}_2 + \ldots + \frac{a_m}{a_1} \underline{x}_m$$

AND WE SEE THAT $\underline{x}$ IS A LINEAR COMBINATION OF POINTS IN $T - \{\underline{x}\}$, IN VIOLATION OF OUR HYPOTHESIS.

SO:

$\boxed{\text{X NOT A LINEAR COMBINATION OF POINTS IN } T - \{x\} \text{ IMPLIES THAT } T \text{ IS LINEARLY INDEPENDENT.}}$

COMBINING, WE HAVE SHOWN T IS LINEARLY INDEPENDENT IF AND ONLY IF $\underline{x}$ IS NOT A LINEAR COMBINATION OF POINTS IN $T - \{x\}$. ✓

2.2-30:

THE PROBLEM TELLS US THAT S IS THE SET OF SOLUTIONS OF THE DIFFERENTIAL EQN. DEFINED ON $C^3[0,\infty)$, WHICH IMPLIES $S \subset C^3[0,\infty)$.

SO ALL WE NEED TO SHOW IS THAT S IS A LINEAR VECTOR SPACE AS DEFINED BY THE PROPERTIES ON 85-86.

① IF $x(t)$ AND $y(t)$ ARE IN S, THEN:

$$\frac{d^3 x(t)}{dt^3} + b\frac{d^2 x(t)}{dt^2} + c\frac{d\,x(t)}{dt} + d\,x(t) = 0$$

$$\frac{d^3 y(t)}{dt^3} + b\frac{d^2 y(t)}{dt^2} + c\frac{d\,y(t)}{dt} + d\,y(t) = 0$$

THEN $a_1 x(t) + a_2 y(t) \in S$ FOR ANY SCALARS a_1 AND a_2, BECAUSE:

$$\frac{d^3}{dt^3}\left(a_1 x(t) + a_2 y(t)\right) + b\frac{d^2}{dt^2}\left(a_1 x(t) + a_2 y(t)\right) + c\frac{d}{dt}\left(a_1 x(t) + a_2 y(t)\right) + d\left(a_1 x(t) + a_2 y(t)\right)$$

$$= a_1\left[\frac{d^3}{dt^3}x(t) + b\frac{d^2}{dt^2}x(t) + c\frac{d}{dt}x(t) + d\,x(t)\right] \to 0$$

$$+ a_2\left[\frac{d^3}{dt^3}y(t) + b\frac{d^2}{dt^2}y(t) + c\frac{d}{dt}y(t) + d\,y(t)\right] \to 0 \quad = 0.$$

THIS PROVES THAT FOR ANY $x(t)$ AND $y(t) \in S$, $x(t) + y(t) \in S$ <u>AND</u> THAT FOR ANY SCALAR a AND $x(t) \in S$, $a\,x(t) \in S$.

② $x(t) + 0 = 0 + x(t) = x(t)$
AND 0 IS CLEARLY A SOLUTION TO THE DIFF. EQ. SO $0 \in S$ AND SATISFIES THE REQUISITE PROPERTIES.

③ FOR ANY $x(t) \in S$, $y(t) = -x(t)$ ALSO SATISFIES THE DIFF. EQN., SO $y(t) = -x(t)$ IS ALSO IN S.

④ THE OTHER PROPERTIES (ASSOCIATIVITY OF VECTOR ADDITION, PROPERTIES OF SCALAR MULTIPLICATION, ETC.) ARE ALL SATISFIED SINCE $C^3[0,\infty)$ IS A VECTOR SPACE, AND $S \subset C^3[0,\infty)$.

2.3-33:

IN A NORMED LINEAR SPACE:

$$\big|\, \|x\| - \|y\| \,\big| \le \|x-y\|$$

PROOF:

THE TRIANGLE INEQUALITY TELLS US:

$\|a+b\| \le \|a\| + \|b\|$ FOR ANY a,b IN OUR LINEAR SPACE.

LET:

$a = x-y \qquad b = y$ WHY? THEN $a+b \Rightarrow x$, $a = x-y$, $b = y$ THE 3 THINGS WE NEED IN OUR EXPRESSION!

THEN WE HAVE:

$$\|x-y+y\| \le \|x-y\| + \|y\|$$

$$\|x\| \le \|x-y\| + \|y\|$$

$$\boxed{\|x\| - \|y\| \le \|x-y\|}$$

THAT GETS US HALF WAY THERE! WE DON'T HAVE AN ABSOLUTE VALUE AROUND $\|x\| - \|y\|$ YET.

NOW LET:

$a = x$ AND $b = y-x$

THEN WE HAVE:

$$\|x+y-x\| \le \|x\| + \|y-x\|$$

$$\|y\| \le \|x\| + \|y-x\|$$

$$\|y-x\| = \|(-1)(x-y)\| = |-1|\,\|x-y\| = \|x-y\|$$

$$\boxed{\|y\| - \|x\| \le \|x-y\|}$$

BUT IF:

$\|y\| - \|x\| \le \|x-y\|$ AND $\|x\| - \|y\| \le \|x-y\|$

WE CAN WRITE:

$$\big|\, \|x\| - \|y\| \,\big| \le \|x-y\| \quad ✓$$

2.3-34:

A FUNCTION f IS CONVEX OVER AN OPEN SET D IF FOR EVERY $s, t \in D$

$$f(\lambda s + (1-\lambda)t) \leq \lambda f(s) + (1-\lambda) f(t)$$

FOR ALL λ SUCH THAT $0 \leq \lambda \leq 1$.

WE NEED TO PROVE THAT A NORM IS CONVEX.

PROOF:

LET $x = \lambda s$ AND $y = (1-\lambda)t$.

BY THE TRIANGLE INEQUALITY, WE HAVE:

$$\|x+y\| \leq \|x\| + \|y\|$$

$$\|\lambda s + (1-\lambda)t\| \leq \|\lambda s\| + \|(1-\lambda)t\| = |\lambda|\,\|s\| + |(1-\lambda)|\,\|t\|$$

FOR $0 \leq \lambda \leq 1$, $1-\lambda \geq 0$ AND $\lambda \geq 0$, SO WE HAVE:

$$\|\lambda s + (1-\lambda)t\| \leq \lambda\|s\| + (1-\lambda)\|t\| \qquad 0 \leq \lambda \leq 1$$

WHICH PROVES ANY NORM IS A CONVEX FUNCTION.

2.3-39:

THE INEQUALITIES OF INTEREST ARE ON P. 96

a) $\|\underline{x}\|_2 \leq \|\underline{x}\|_1 \leq \sqrt{n}\,\|\underline{x}\|_2$ ⇐ TWO CASES EACH

b) $\|\underline{x}\|_\infty \leq \|\underline{x}\|_2 \leq \sqrt{n}\,\|\underline{x}\|_\infty$

c) $\|\underline{x}\|_\infty \leq \|\underline{x}\|_1 \leq n\,\|\underline{x}\|_\infty$

WE NEED TO FIND VECTORS IN $\mathbb{R}^n$ WHICH ACHIEVE EQUALITY.

$\|\underline{x}\|_2 = \|\underline{x}\|_1$? HOW ABOUT A VECTOR $\underline{x}$ THAT ONLY HAS 1 NON-ZERO COMPONENT?

(a-1)

$$\boxed{\underline{x} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \begin{array}{l} \|\underline{x}\|_2 = 1 \\ \|\underline{x}\|_1 = 1 \end{array}}$$

✓

WHAT ABOUT $\|\underline{x}\|_1 = \sqrt{n}\,\|\underline{x}\|_2$?

TRY:

(a-2)

$$\boxed{\underline{x} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \qquad \begin{array}{l} \|\underline{x}\|_2 = \sqrt{1+1+\dots+1} = \sqrt{n} \\ \|\underline{x}\|_1 = 1+1+\dots+1 = n \end{array} \quad \text{SO } \|\underline{x}\|_1 = \sqrt{n}\,\|\underline{x}\|_2}$$

✓

(b-1) $\|\underline{x}\|_\infty = \|\underline{x}\|_2$

LET'S TRY THE VECTOR $\underline{x}$ THAT ONLY HAS 1 NON-ZERO COMPONENT AGAIN.

$$\boxed{\underline{x} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \begin{array}{l} \|\underline{x}\|_\infty = 1 \\ \|\underline{x}\|_2 = 1 \end{array}}$$

✓

(b-2) $\|\underline{x}\|_2 = \sqrt{n}\,\|\underline{x}\|_\infty$

TRY

$$\boxed{\underline{x} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \text{ AGAIN.} \qquad \begin{array}{l} \|\underline{x}\|_2 = \sqrt{1+1+\dots+1} = \sqrt{n} \\ \|\underline{x}\|_\infty = 1 \quad \text{SO } \sqrt{n}\,\|\underline{x}\|_\infty = \sqrt{n} \end{array}}$$

✓

(c-1) $\|\underline{x}\|_\infty = \|\underline{x}\|_1$

ONCE AGAIN, HOW ABOUT $\underline{x} = \begin{pmatrix}1\\0\\0\\ \vdots\\0\end{pmatrix}$ $\quad \|\underline{x}\|_\infty = 1$

$\|\underline{x}\|_1 = 1$

✓

(c-2) $\|\underline{x}\|_1 = n\|\underline{x}\|_\infty$

AND NOW TRY $\underline{x} = \begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix}$ AGAIN.

$\underline{x} = \begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix}$ $\quad \|\underline{x}\|_1 = n$

$\|\underline{x}\|_\infty = 1 \quad$ SO $n\|\underline{x}\|_\infty = n$

✓

2.4-42:

IN ORDER TO BE A LEGITIMATE INNER PRODUCT, $\langle f,g\rangle$ MUST SATISFY THE FOLLOWING:

① $\langle f,g\rangle = \overline{\langle g,f\rangle}$

② $\langle \alpha f,g\rangle = \alpha\langle f,g\rangle$

③ $\langle f_1+f_2,g\rangle = \langle f_1,g\rangle + \langle f_2,g\rangle$

④ $\langle f,f\rangle > 0$ IF $f\neq 0$, AND $\langle f,f\rangle = 0$ IFF $f=0$.

(i) $\langle f,g\rangle = \int_0^1 f'(t)g'(t)\,dt + f(0)g(0)$

CHECK EACH OF THE PROPERTIES:

① $\overline{\langle g,f\rangle} = \overline{\int_0^1 g'(t)f'(t)\,dt + g(0)f(0)} = \int_0^1 f'(t)g'(t)\,dt + f(0)g(0)$

SINCE f,g ARE REAL $= \langle f,g\rangle$ ✓

② $\langle \alpha f,g\rangle = \int_0^1 \alpha f'(t)g'(t)\,dt + \alpha f(0)g(0) = \alpha\left[\int_0^1 f'(t)g'(t)\,dt + f(0)g(0)\right]$ ✓

③ $\langle f_1+f_2,g\rangle = \int_0^1 (f_1+f_2)'(t)g'(t)\,dt + (f_1+f_2)(0)g(0)$

$= \int_0^1 f_1'(t)g'(t)\,dt + f_1(0)g(0) + \int_0^1 f_2'(t)g'(t)\,dt + f_2(0)g(0)$

$= \langle f_1,g\rangle + \langle f_2,g\rangle$ ✓

④ $\langle f,f\rangle = \int_0^1 f'(t)f'(t)\,dt + f(0)f(0) > 0$ IF $f\neq 0$ SINCE f IS REAL (SO $[f'(t)]^2$ IS ≥ 0 AND $[f(0)]^2 > 0$

$= 0$ IF $f=0$ ✓

SO (i) IS A VALID INNER PRODUCT OVER THE SPACE.

(ii) $\langle f, g \rangle = \int_0^1 f'(t) g'(t)\, dt$

CHECK EACH OF THE PROPERTIES:

① $\overline{\langle g, f \rangle} = \overline{\int_0^1 g'(t) f'(t)\, dt} = \int_0^1 f'(t) g'(t)\, dt$ SINCE f, g ARE REAL

$= \langle g, f \rangle$ ✓

② $\langle \alpha f, g \rangle = \int_0^1 \alpha f'(t) g'(t)\, dt = \alpha \int_0^1 f'(t) g'(t)\, dt = \alpha \langle f, g \rangle$ ✓

③ $\langle f_1 + f_2, g \rangle = \int_0^1 (f_1 + f_2)'(t) g'(t)\, dt = \int_0^1 f_1'(t) g'(t)\, dt + \int_0^1 f_2'(t) g'(t)\, dt$

$= \langle f_1, g \rangle + \langle f_2, g \rangle$ ✓

④ $\langle f, f \rangle = \int_0^1 f'(t) f'(t)\, dt \geq 0$ IF $f \neq 0$

$\Uparrow$

COULD BE EQUAL TO ZERO FOR NON-ZERO f! (FOR EXAMPLE, CONSTANT f HAS $f'(t) = 0$)!

$\boxed{\text{SO (ii) ISN'T A VALID INNER PRODUCT!}}$

2.5-43:

(a) FOR AN ~~INDUCED~~ NORM $\|\cdot\|$ WE KNOW THAT

$$\|\underline{x}\|^2 = \langle \underline{x}, \underline{x} \rangle$$

SO:

$$\|\underline{x}+\underline{y}\|^2 = \langle \underline{x}+\underline{y}, \underline{x}+\underline{y} \rangle = \langle \underline{x}, \underline{x} \rangle + \langle \underline{x}, \underline{y} \rangle + \langle \underline{y}, \underline{x} \rangle + \langle \underline{y}, \underline{y} \rangle$$

AND:

$$\|\underline{x}-\underline{y}\|^2 = \langle \underline{x}-\underline{y}, \underline{x}-\underline{y} \rangle = \langle \underline{x}, \underline{x} \rangle - \langle \underline{x}, \underline{y} \rangle - \langle \underline{y}, \underline{x} \rangle + \langle \underline{y}, \underline{y} \rangle$$

SO:

$$\|\underline{x}+\underline{y}\|^2 + \|\underline{x}-\underline{y}\|^2 = \langle \underline{x}, \underline{x} \rangle + \cancel{\langle \underline{x}, \underline{y} \rangle} + \cancel{\langle \underline{y}, \underline{x} \rangle} + \langle \underline{y}, \underline{y} \rangle + \langle \underline{x}, \underline{x} \rangle - \cancel{\langle \underline{x}, \underline{y} \rangle} - \cancel{\langle \underline{y}, \underline{x} \rangle} + \langle y, y \rangle$$

$$\|\underline{x}+\underline{y}\|^2 + \|\underline{x}-\underline{y}\|^2 = 2\langle \underline{x}, \underline{x} \rangle + 2\langle \underline{y}, \underline{y} \rangle = 2\|\underline{x}\|^2 + 2\|\underline{y}\|^2 \checkmark$$

(b) SHOW THAT: $\langle \underline{x}, \underline{y} \rangle = \dfrac{\|\underline{x}+\underline{y}\|^2 - \|\underline{x}-\underline{y}\|^2}{4}$ OVER A REAL VECTOR SPACE.

PROOF:

$$\|\underline{x}+\underline{y}\|^2 - \|\underline{x}-\underline{y}\|^2 = \cancel{\langle \underline{x}, \underline{x} \rangle} + \langle \underline{x}, \underline{y} \rangle + \langle \underline{y}, \underline{x} \rangle + \cancel{\langle \underline{y}, \underline{y} \rangle} - \cancel{\langle \underline{x}, \underline{x} \rangle} + \langle \underline{x}, \underline{y} \rangle + \langle \underline{y}, \underline{x} \rangle - \cancel{\langle \underline{y}, \underline{y} \rangle}$$

$$\|\underline{x}+\underline{y}\|^2 - \|\underline{x}-\underline{y}\|^2 = 2\langle \underline{x}, \underline{y} \rangle + 2\langle \underline{y}, \underline{x} \rangle$$

BUT THE VECTOR SPACE IS REAL, SO $\langle \underline{y}, \underline{x} \rangle = \overline{\langle \underline{x}, \underline{y} \rangle} = \langle \underline{x}, \underline{y} \rangle$

SO:

$$\|\underline{x}+\underline{y}\|^2 - \|\underline{x}-\underline{y}\|^2 = 4\langle \underline{x}, \underline{y} \rangle$$

OR:

$$\langle \underline{x}, \underline{y} \rangle = \frac{\|\underline{x}+\underline{y}\|^2 - \|\underline{x}-\underline{y}\|^2}{4} \checkmark$$