ECEn 671 | SOLUTIONS HOMEWORK #2

2.1-19: NO, IT IS NOT NECESSARILY TRUE.

CONSIDER THE FOLLOWING EXAMPLE:

LET $a_n = (-1)^n$ AND $b_n = -(-1)^n$.

THEN: $\limsup_{n \to \infty} a_n = 1$

$\limsup_{n \to \infty} b_n = 1$

BUT:

$$\limsup_{n \to \infty} (a_n + b_n) = \limsup_{n \to \infty} (0) = 0 \neq 1 + 1$$

2.1-20: $\{x_n\}$ IS A SEQUENCE SUCH THAT

$$d(x_{n+1}, x_n) < Cr^n \quad \text{FOR } 0 < r < 1$$

NOT $\leq$ LIKE IN PROBLEM. TYPO!

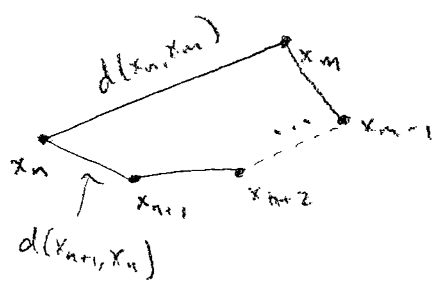
PROVE THAT $\{x_n\}$ IS A CAUCHY SEQUENCE.

PROOF:

RECALL THAT A SEQUENCE IS CAUCHY IF, FOR ANY $\varepsilon > 0$, THERE IS AN $N > 0$ SUCH THAT

$$d(x_n, x_m) < \varepsilon \quad \text{FOR EVERY } m, n > N.$$

WITHOUT LOSS OF GENERALITY, ASSUME $m > n$, SO THAT $m = n + k$ FOR $k \geq 1$.



BY REPEATED APPLICATION OF THE TRIANGLE INEQUALITY, WE HAVE:

$$d(x_n, x_m) \leq d(x_{n+1}, x_n) + d(x_{n+2}, x_{n+1}) + \ldots + d(x_m, x_{m-1})$$

$$d(x_n, x_m) \leq Cr^n + Cr^{n+1} + \ldots + Cr^{m-1}$$

USING $m = n+k$:

$$d(x_n, x_m) \le Cr^n + Cr^{n+1} + \dots + Cr^{n+k-1}$$

$$d(x_n, x_m) \le Cr^n(1 + r + \dots + r^{k-1})$$

$$d(x_n, x_m) \le Cr^n\left(\frac{1-r^k}{1-r}\right)$$

FINITE GEOMETRIC SUM W/ $0<r<1$

$$d(x_n, x_m) \le Cr^n\left(\frac{1-r^k}{1-r}\right) < Cr^n\left(\frac{1}{1-r}\right)$$

SO:

$$d(x_n, x_m) < Cr^n\left(\frac{1}{1-r}\right)$$ FOR ANY m, n W/ $m>n$.

FOR n SUFFICIENTLY LARGE, WE CAN MAKE $Cr^n\left(\frac{1}{1-r}\right)$ SMALLER THAN ANY $\varepsilon > 0$.

THUS, THE SEQUENCE IS CAUCHY.

2.1-22: SHOW THAT IF A SEQUENCE $\{x_n\}$ IS CONVERGENT, IT IS CAUCHY.

PROOF:

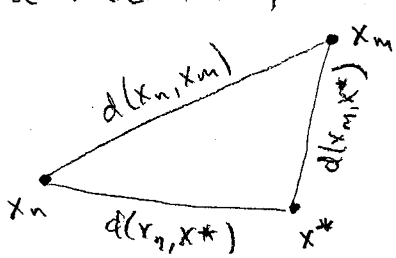
IF $\{x_n\}$ IS CONVERGENT, THEN THERE IS AN n_0 SUCH THAT $d(x_n, x^*) < \delta$ FOR EVERY $\delta > 0$ AND $n > n_0$ FOR SOME FIXED VALUE x^*. (DEFINITION OF CONVERGENCE.)

LET N BE SUCH THAT:

$d(x_n, x^*) < \frac{\varepsilon}{2}$ FOR $n > N$ FOR ANY $\varepsilon > 0$.

LET $m > N$ AND $n > N$. BY THE TRIANGLE INEQUALITY, WE HAVE:

$$d(x_n, x_m) \le \underbrace{d(x_n, x^*)}_{<\frac{\varepsilon}{2}} + \underbrace{d(x_m, x^*)}_{<\frac{\varepsilon}{2}}$$



$$d(x_n, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

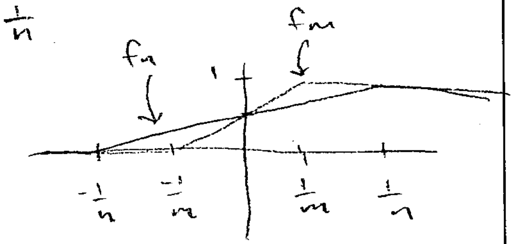
SO:

$$d(x_n, x_m) < \varepsilon \quad \text{FOR ANY } n, m > N.$$

THIS HOLDS FOR ANY $\varepsilon > 0$, SO $\{x_n\}$ IS CAUCHY.

2.1-24: FROM (2.6), WE HAVE:

$$f_n(t) = \begin{cases} 0, & t < -\frac{1}{n} \\ nt/2 + 1/2, & -\frac{1}{n} \le t \le \frac{1}{n} \\ 1, & t > \frac{1}{n} \end{cases}$$



FOR $m > n$ AND $t \geq 0$, WE HAVE:

$$|f_m(t) - f_n(t)| = \begin{cases} \left|\frac{mt}{2} + \frac{1}{2} - \frac{nt}{2} - \frac{1}{2}\right|, & 0 \le t \le \frac{1}{m} \\ \left|1 - \frac{nt}{2} - \frac{1}{2}\right|, & \frac{1}{m} < t \le \frac{1}{n} \\ 0, & t > \frac{1}{n} \end{cases}$$

(THE FUNCTION IS SYMMETRIC ABOUT $t = 0$, SO WE ONLY NEED LOOK AT $t \geq 0$)

$$|f_m(t) - f_n(t)| = \begin{cases} \frac{(m-n)t}{2}, & 0 \le t \le \frac{1}{m} \\ \frac{1}{2} - \frac{nt}{2}, & \frac{1}{m} < t \le \frac{1}{n} \\ 0, & t > \frac{1}{n} \end{cases}$$

MAX OCCURS AT $t = \frac{1}{m}$, SO:

$$\sup_t |f_m(t) - f_n(t)| = \frac{1}{2} - \frac{n}{2m} = d_\infty(f_n, f_m).$$

FOR ANY LARGE n, m, WE CAN SEE THAT $\frac{1}{2} - \frac{n}{2m}$ IS NOT NECESSARILY SMALL. IN FACT, FOR ANY LARGE n, SAY $n = N$, WE CAN MAKE $\frac{1}{2} - \frac{n}{2m}$ AS CLOSE TO $\frac{1}{2}$ AS WE LIKE BY CHOOSING m SUFFICIENTLY LARGE. THUS, f_n IS NOT CAUCHY.

2.7-47:

$x_1(t) = 3t^2 - 1 \qquad x_2(t) = 5t^3 - 3t \qquad x_3(t) = 2t^2 - t$

$$\langle f, g \rangle \equiv \int_{-1}^{1} f(t)g(t)\,dt.$$

FIRST COMPUTE:

$$\langle x_1(t), x_2(t) \rangle = \int_{-1}^{1} (3t^2-1)(5t^3-3t)\,dt$$

$$\langle x_1(t), x_3(t) \rangle = \int_{-1}^{1} (3t^2-1)(2t^2-t)\,dt$$

$$\langle x_2(t), x_3(t) \rangle = \int_{-1}^{1} (5t^3-3t)(2t^2-t)\,dt$$

AND:

$$\|x_1(t)\| = \left(\int_{-1}^{1} (3t^2-1)^2\,dt\right)^{\frac{1}{2}}$$

$$\|x_2(t)\| = \left(\int_{-1}^{1} (5t^3-3t)^2\,dt\right)^{\frac{1}{2}}$$

$$\|x_3(t)\| = \left(\int_{-1}^{1} (2t^2-t)^2\,dt\right)^{\frac{1}{2}}$$

THEN:

$$\theta_{12} = \cos^{-1}\left(\frac{\langle x_1(t), x_2(t)\rangle}{\|x_1(t)\|\,\|x_2(t)\|}\right)$$

$$\theta_{13} = \cos^{-1}\left(\frac{\langle x_1(t), x_3(t)\rangle}{\|x_1(t)\|\,\|x_3(t)\|}\right)$$

$$\theta_{23} = \cos^{-1}\left(\frac{\langle x_2(t), x_3(t)\rangle}{\|x_2(t)\|\,\|x_3(t)\|}\right)$$

BY MATLAB: (SEE NEXT PAGE FOR CODE)

9/22/10 2:42 PM MATLAB Command Window

```
Angle between x1(t) and x2(t) is 1.571 radians or 90 deg.
Angle between x1(t) and x3(t) is 0.9763 radians or 55.94 deg.
Angle between x2(t) and x3(t) is 1.571 radians or 90 deg.
```

$\boxed{\text{$x_1(t)$ AND $x_2(t)$ ARE ORTHOGONAL, AND $x_2(t)$ AND $x_3(t)$ ARE ORTHOGONAL.}}$

```
%   Create t variable on interval [-1, 1]
dt = 0.00001;
t = -1:dt:1;
ln = length(t)-1;

%   Create vectors x1, x2, and x3
x1 = 3*t.^2 - 1;
x2 = 5*t.^3 - 3*t;
x3 = 2*t.^2 - t;

%   Compute norms
tmp = x1.*x1*dt;
tmp = tmp(1:ln);
norm_x1 = sqrt(sum(tmp));

tmp = x2.*x2*dt;
tmp = tmp(1:ln);
norm_x2 = sqrt(sum(tmp));

tmp = x3.*x3*dt;
tmp = tmp(1:ln);
norm_x3 = sqrt(sum(tmp));

%   Compute the inner products
tmp = x1.*x2*dt;
tmp = tmp(1:ln);
x1_dot_x2 = sum(tmp);

tmp = x1.*x3*dt;
tmp = tmp(1:ln);
x1_dot_x3 = sum(tmp);

tmp = x2.*x3*dt;
tmp = tmp(1:ln);
x2_dot_x3 = sum(tmp);

%   Compute the angles
ang_x1_x2 = acos(x1_dot_x2/(norm_x1*norm_x2));
ang_x1_x3 = acos(x1_dot_x3/(norm_x1*norm_x3));
ang_x2_x3 = acos(x2_dot_x3/(norm_x2*norm_x3));

%   Output the results
disp(sprintf('Angle between x1(t) and x2(t) is %0.4g radians or %0.4g deg.',
ang_x1_x2, ang_x1_x2/pi*180));
disp(sprintf('Angle between x1(t) and x3(t) is %0.4g radians or %0.4g deg.',
ang_x1_x3, ang_x1_x3/pi*180));
disp(sprintf('Angle between x2(t) and x3(t) is %0.4g radians or %0.4g deg.',
ang_x2_x3, ang_x2_x3/pi*180));
```

CODE FOR 2.7-47

SHOULD REALLY WRITE A FUNCTION FOR THESE, BUT I WAS LAZY... :)

2.7-49:

SHOW THAT ORTHOGONALITY OF $\{\underline{P}_1, \underline{P}_2, \ldots, \underline{P}_m\}$ IMPLIES LINEAR INDEPENDENCE.

FOR $\underline{P}_i$ NON-ZERO!

PROOF:

IF $\{\underline{P}_1, \underline{P}_2, \ldots, \underline{P}_m\}$ ARE ORTHOGONAL, THEN

$\langle P_i, P_j \rangle = 0$ FOR $i \neq j$. (GIVEN)

NOW SUPPOSE THE SET IS LINEARLY DEPENDENT (THIS IS GOING TO BE A PROOF BY CONTRADICTION).

THEN THERE IS A SET OF COEFFICIENTS $a_1, a_2, \ldots, a_m$, NOT ALL ZERO, FOR WHICH:

$$a_1 \underline{P}_1 + a_2 \underline{P}_2 + \ldots + a_m \underline{P}_m = \underline{0}.$$

NOW LET'S TAKE THE INNERPRODUCT OF THIS EQUATION WITH EACH OF THE $\underline{P}_i$ VECTORS:

ONLY THIS TERM SURVIVES!

$$\langle \underbrace{a_1 \underline{P}_1 + a_2 \underline{P}_2 + \ldots + a_m \underline{P}_m}_{= \underline{0}}, \underline{P}_1 \rangle = a_1 \langle \underline{P}_1, \underline{P}_1 \rangle = 0$$

SINCE THE INNER PRODUCT OF $\underline{0}$ WITH ANYTHING IS 0!

WE KNOW THAT $\langle \underline{P}_1, \underline{P}_1 \rangle > 0$,

SINCE THE $\underline{P}_i$ ARE NON-ZERO,

SO:

$a_1 \langle \underline{P}_1, \underline{P}_1 \rangle = 0$ IMPLIES $\boxed{a_1 = 0.}$

REPEAT WITH $\underline{P}_2$:

$$\langle a_1 \underline{P}_1 + a_2 \underline{P}_2 + \ldots + a_m \underline{P}_m, \underline{P}_2 \rangle = a_2 \langle \underline{P}_2, \underline{P}_2 \rangle = 0 \quad \text{SO} \quad \boxed{a_2 = 0}$$

AND SO FORTH. THUS, IF $\{\underline{P}_i\}$ ARE ORTHOGONAL, ALL OF THE a_i ARE ZERO, AND $\{\underline{P}_i\}$ IS THEREFORE LINEARLY INDEPENDENT.

2.13-68:

$A = U \Sigma V^H$ WHERE $\underbrace{U^H U = I \quad V^H V = I}$

U AND V ARE "UNITARY"

NOTICE THAT THESE IMPLY:

$U^H = U^{-1}$

AND $V^H = V^{-1}$

Σ IS DIAGONAL WITH REAL VALUES.

SHOW THAT $P_A = P_U$.

$P_A \equiv A(A^H A)^{-1} A^H$ (P. 116 IN BOOK, OR NOTES)

$P_U \equiv U(U^H U)^{-1} U^H$

SO:

$$P_A = A(A^H A)^{-1} A^H = U\Sigma V^H\left[(U\Sigma V^H)^H U \Sigma V^H\right]^{-1}(U\Sigma V^H)^H$$

$$= U\Sigma V^H \left(V \Sigma^H \underbrace{U^H U}_{I} \Sigma V^H\right)^{-1} V \Sigma^H U^H$$

$$= U\Sigma V^H \left(V \Sigma^H \Sigma V^H\right)^{-1} V \Sigma^H U^H$$

$$= U\Sigma V^H \underbrace{(V^H)^{-1}}_{V} (\Sigma^H \Sigma)^{-1} \underbrace{V^{-1} V}_{I} \Sigma^H U^H$$

$$= U \Sigma \underbrace{V^H V}_{I} (\Sigma^H \Sigma)^{-1} \Sigma^H U^H$$

$$= U \Sigma (\Sigma^H \Sigma)^{-1} \Sigma^H U^H = U \underbrace{\Sigma \Sigma^{-1}}_{I} \underbrace{(\Sigma^H)^{-1} \Sigma^H}_{I} U^H = U U^H = I \checkmark$$

AND:

$$P_U = U \underbrace{(U^H U)}_{I}^{-1} U^H = U U^H = I \checkmark$$

2.13-71:

LET $P_1, P_2, \ldots, P_m$ BE ORTHOGONAL PROJECTIONS

(MEANING $P_i^2 = P_i$, AND P_i IS SYMMETRIC (OR HERMITIAN))

P.3 OF LECTURE 5 NOTES
OR THEOREM 2.8 IN BOOK

ALSO GIVEN IS $P_i P_j = 0$ FOR $i \neq j$.

SHOW THAT $Q = P_1 + P_2 + \ldots + P_m$ IS AN ORTHOGONAL PROJECTION.

PROOF:

Q IS AN ORTHOGONAL PROJECTION MATRIX IF AND ONLY IF $Q^2 = Q$ AND Q IS HERMITIAN.

DOES $Q^2 = Q$?

CROSS TERMS ARE 0!

$$Q^2 = (P_1 + P_2 + \ldots + P_m)^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} P_i P_j = \sum_{i=1}^{m} P_i^2 = \sum_{i=1}^{m} P_i = Q \checkmark$$

IS Q HERMITIAN?

$$Q^H = (P_1 + P_2 + \ldots + P_m)^H = P_1^H + P_2^H + \ldots + P_m^H = P_1 + P_2 + \ldots + P_m = Q \checkmark$$

SINCE P_i IS HERMITIAN

SO:

$Q^2 = Q$

AND $Q^H = Q$, SO Q IS AN ORTHOGONAL PROJECTION.

2.15-77:

THIS IS PAINFUL TO DO BY HAND, BUT LET'S SET IT UP BY HAND AND THEN LET MATLAB CRUNCH THE NUMBERS.

BY THE GRAM-SCHMIDT PROCESS:

$$q_1 = \frac{p_1}{\|p_1\|}$$

$$e_2 = p_2 - \langle p_2, q_1 \rangle q_1$$

$$q_2 = \frac{e_2}{\|e_2\|}$$

$$e_3 = p_3 - \langle p_3, q_1 \rangle q_1 - \langle p_3, q_2 \rangle q_2$$

$$q_3 = \frac{e_3}{\|e_3\|}$$

$$e_4 = p_4 - \langle p_4, q_1 \rangle q_1 - \langle p_4, q_2 \rangle q_2 - \langle p_4, q_3 \rangle q_3$$

$$q_4 = \frac{e_4}{\|e_4\|}$$

WE COULD STRUCTURE OUR GRAM-SCHMIDT CODE MORE SUCCINCTLY AND GENERALLY AS DESCRIBED BRIEFLY IN THE BOOK, BUT FOR THIS SMALL CASE WE'LL WRITE MATLAB CODE THAT SIMPLY FOLLOWS THE OUTLINE ABOVE.

TO SIMPLIFY OUR MATLAB CODE, WE'LL REPRESENT EACH FUNCTION AS A VECTOR OF LENGTH 4, WHERE THE VALUES MAP AS FOLLOWS:

VALUE 1 $\Rightarrow f(t)$ FOR $0 \le t < \frac{1}{4}$

VALUE 2 $\Rightarrow f(t)$ FOR $\frac{1}{4} \le t < \frac{1}{2}$

VALUE 3 $\Rightarrow f(t)$ FOR $\frac{1}{2} \le t < 3/4$

VALUE 4 $\Rightarrow f(t)$ FOR $3/4 \le t \le 1$

IN THIS REPRESENTATION:

$$P_1 = (1, 1, 1, 1)^T \quad P_2 = (1, 1, -1, 1)^T$$

$$P_3 = (1, 1, -1, -1)^T \text{ AND } P_4 = (1, -1, 1, 1)^T$$

WE CAN EASILY COMPUTE $\langle f, g \rangle$ AS:

$$\langle f, g \rangle = g^T f \cdot \frac{1}{4}$$ ⇐ CONVINCE YOURSELF THAT THIS GIVES $\int_0^1 f(t) g(t) dt$.

AND THEN:

$$\|f\| = \mathrm{sqrt}(f^T f)$$

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```
%
%   Gram-Schmidt code for 2.15-77
%

clear;
close all;

%   Set up p1 - p4
p1 = [1; 1; 1; 1];
p2 = [1; 1; -1; 1];
p3 = [1; 1; -1; -1];
p4 = [1; -1; 1; 1];

%   Compute q1, which is just a normalized p1
%   (turns out p1 is already normalized...)
q1 = p1/sqrt(p1'*p1*1/4);

%   Now find e2
e2 = p2 - (q1'*p2*1/4)*q1;

%   Normalize to find q2
q2 = e2/sqrt(e2'*e2*1/4);

%   Find e3
e3 = p3 - (q1'*p3*1/4)*q1 - (q2'*p3*1/4)*q2;

%   Normalize to find q3
q3 = e3/sqrt(e3'*e3*1/4);

%   And finally fine e4
e4 = p4 - (q1'*p4*1/4)*q1 - (q2'*p4*1/4)*q2 - (q3'*p4*1/4)*q3;

%   Normalize to get q4
q4 = e4/sqrt(e4'*e4*1/4);


%   Create some basis vectors to make it easy to plot :)
t = 0:0.001:1;
ln = length(t);
b = zeros(4, ln);
b(1,1:floor(ln/4)) = 1;
b(2,floor(ln/4)+1:floor(ln/2)) = 1;
b(3,floor(ln/2)+1:floor(3*ln/4)) = 1;
b(4,floor(3*ln/4)+1:ln) = 1;

%   And plot
subplot(2,2,1);
plot(t, q1'*b);
title('q1(t)');

subplot(2,2,2);
plot(t, q2'*b);
title('q2(t)');

subplot(2,2,3);
plot(t, q3'*b);
title('q3(t)');

subplot(2,2,4);
plot(t, q4'*b);
title('q4(t)');
```

RESULTS