3.14-27:

GIVEN HOW THE PROBLEM IS SET UP (INTEGRALS W/ LOWER BOUND OF 0 ON $y(t)$ AND $h(t)$) ASSUME CAUSALITY ($h(t)=0, t<0$) AND INITIAL REST ($y(t)=0, t<0$).

WE WANT TO SHOW:

$$\int_0^T y(t)\,dt = x(t) * k(t)\Big|_{t=T} \qquad \text{WHERE } k(t) = \int_0^t h(\tau)\,d\tau = \underbrace{\int_{-\infty}^t h(\tau)\,d\tau}_{\text{CAUSALITY!}}$$

FIRST LOOK AT LEFT HAND SIDE:

$$\int_0^T y(t)\,dt = \underbrace{\int_{-\infty}^T y(t)\,dt}_{\text{INITIAL REST!}} = \int_{-\infty}^T x(t) * h(t)\,dt = \int_{-\infty}^T \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\,d\tau\,dt$$

$$= \boxed{\int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{T} h(t-\tau)\,dt\,d\tau}$$

NOW LOOK AT RIGHT HAND SIDE:

$$x(t) * k(t)\Big|_{t=T} = \int_{-\infty}^{\infty} x(\tau)\, k(t-\tau)\,d\tau\Big|_{t=T}$$

$$k(t-\tau) = \int_{-\infty}^{t-\tau} h(s)\,ds$$

NEED A NEW DUMMY VARIABLE, SINCE I'M ALREADY USING τ IN LIMITS!

$$= \int_{-\infty}^{\infty} x(\tau) \int_{s=-\infty}^{s=t-\tau} h(s)\,ds\,d\tau$$

$\Leftarrow$ DO CHANGE OF VARIABLES $s = u - \tau$ $\quad ds = du$

SO THIS INTEGRAL LOOKS LIKE THE L.H.S. ABOVE

$$= \int_{-\infty}^{\infty} x(\tau) \int_{u-\tau=-\infty}^{u-\tau=t-\tau} h(u-\tau)\,du\,d\tau$$

$$= \boxed{\int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{t} h(u-\tau)\,du\,d\tau}$$

SAME AS L.H.S.! ✓

3.14-28:

$h(t) = 3e^{-2t} + 4e^{-5t}$ FOR $t>0$ CAUSALITY AND INITIAL REST ASSUMED.

WE WANT TO FIND AN INPUT $x(t)$ THAT SATISFIES THE CONSTRAINTS:

$$y(2) = 2 \quad \text{AND} \quad \int_0^2 y(t)\,dt = 3 \quad \text{, WHERE } y(t) = x(t) * h(t)$$

WITH MINIMUM INPUT ENERGY $\|x(t)\|^2$. ← $= \int_0^2 |x(t)|^2 dt$

WE WANT TO CAST THIS AS A DUAL APPROXIMATION PROBLEM, WHICH MEANS WE WANT TO GET THE TWO CONSTRAINTS IN THE FORM:

$$\langle x(t), y_1(t)\rangle = a_1 \quad \text{OR} \quad \int_0^{2 \leftarrow \text{WE ONLY CARE ABOUT } t\in[0,2]!} x(t)\,y_1(t)\,dt = a_1$$

$$\langle x(t), y_2(t)\rangle = a_2 \qquad \int_0^2 x(t)\,y_2(t)\,dt = a_2$$

REMEMBER THAT:

$$y(t) = x(t) * h(t) = \int_0^t x(\tau)\,h(t-\tau)\,d\tau$$

t ← SINCE $h(t)=0,\ t<0$ (CAUSALITY)

0 ← SINCE $x(t)=0,\ t<0$ (INITIAL REST)

SO OUR FIRST CONSTRAINT IS:

$$y(2) = \int_0^2 x(\tau)\,h(2-\tau)\,d\tau = 2$$

$$\boxed{\langle x(t), y_1(t)\rangle = 2} \quad \text{WHERE } y_1(t) = h(2-t) = 3e^{-2(2-t)} + 4e^{-5(2-t)}$$

HOW ABOUT CONSTRAINT #2?

$$\int_0^2 y(t)\,dt = x(t) * k(t)\Big|_{t=2} \quad \text{WHERE } k(t) = \int_0^t h(\tau)\,d\tau$$

⇑ FROM LAST PROBLEM!

SO:

$$k(t) = \int_0^t \left(3e^{-2\tau} + 4e^{-5\tau}\right) d\tau = \left(-\frac{3}{2}e^{-2\tau} - \frac{4}{5}e^{-5\tau}\right)\Big|_0^t$$

$$k(t) = -\frac{3}{2}e^{-2t} - \frac{4}{5}e^{-5t} + \frac{3}{2} + \frac{4}{5} = \frac{23}{10} - \frac{3}{2}e^{-2t} - \frac{4}{5}e^{-5t}$$

AND:

$$\int_0^2 y(t)\,dt = x(t) * k(t)\Big|_{t=2} = \int_{\underset{\text{SINCE } x(t)=0,\ t<0}{0}}^{t\ \leftarrow \text{SINCE } k(t)=0,\ t<0} x(\tau)\,k(t-\tau)\,d\tau \Big|_{t=2}$$

$$= \int_0^2 x(\tau)\,k(2-\tau)\,d\tau = 3$$

$$\langle x(t), y_2(t)\rangle = 3 \quad \text{WHERE} \quad y_2(t) = k(2-t) = \frac{23}{10} - \frac{3}{2}e^{-2(2-t)} - \frac{4}{5}e^{-5(2-t)}$$

NOW THAT WE HAVE OUR CONSTRAINTS, BY THE <u>DUAL-APPROXIMATION THEOREM</u>: (THEOREM 3.4 IN BOOK)

$$x(t) = c_1 y_1(t) + c_2 y_2(t)$$

WHERE c_1, c_2 ARE GIVEN BY:

$$\begin{bmatrix} \langle y_1, y_1\rangle & \langle y_2, y_1\rangle \\ \langle y_1, y_2\rangle & \langle y_2, y_2\rangle \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

NOW WE NEED TO FIND THESE AND SOLVE FOR c_1 AND c_2! :)

$$\langle y_1, y_1\rangle = \int_0^2 \left(3e^{-2(2-t)} + 4e^{-5(2-t)}\right)^2 dt$$

$$\langle y_1, y_2\rangle = \langle y_2, y_1\rangle = \int_0^2 \left(3e^{-2(2-t)} + 4e^{-5(2-t)}\right)\left(\frac{23}{10} - \frac{3}{2}e^{-2(2-t)} - \frac{4}{5}e^{-5(2-t)}\right) dt$$

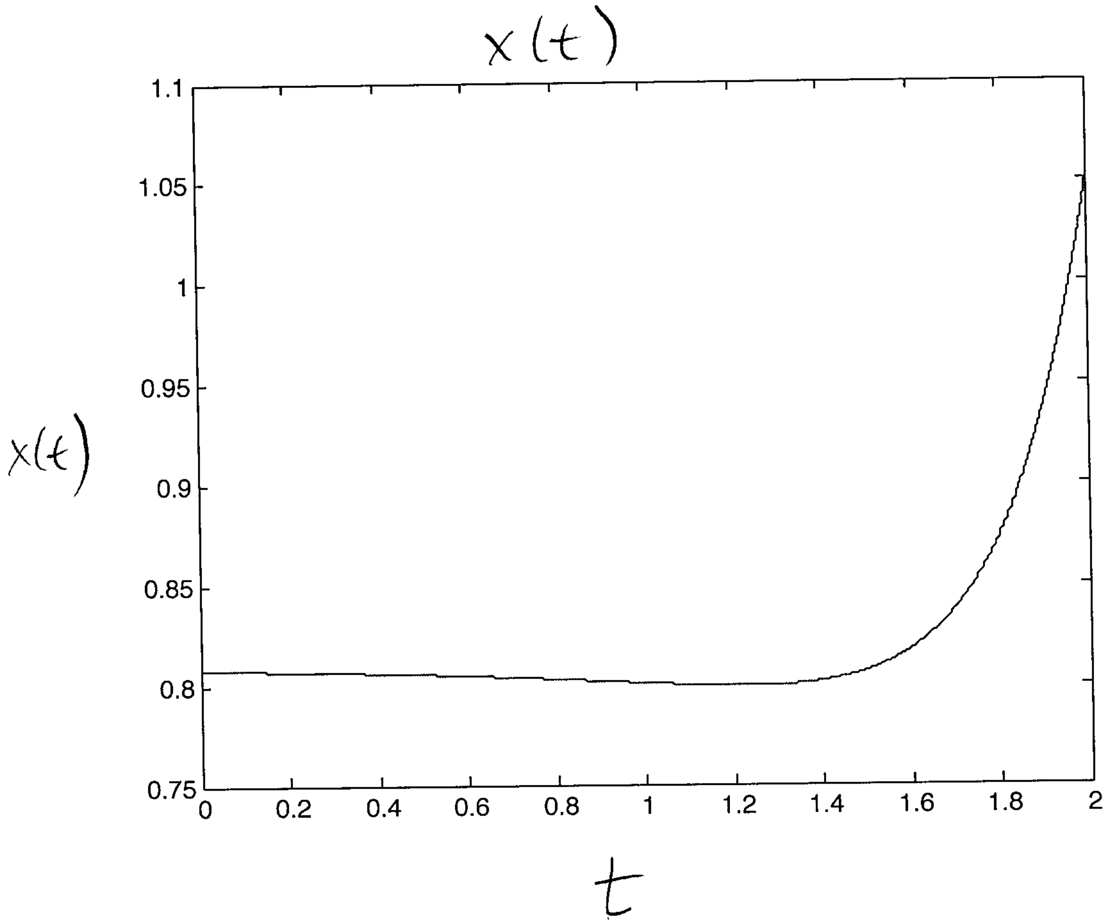
$$\langle y_2, y_2\rangle = \int_0^2 \left(\frac{23}{10} - \frac{3}{2}e^{-2(2-t)} - \frac{4}{5}e^{-5(2-t)}\right)^2 dt$$

$\boxed{\text{LOOKS LIKE A JOB FOR MATLAB! :)}}$

$$C = \begin{pmatrix} 0.15 \\ 0.35 \end{pmatrix}$$

so:

$$x(t) = 0.15\, y_1(t) + 0.35\, y_2(t)$$



3.14-29:

$h[t] = (0.2)^t + 3(0.4)^t$ FOR $t \geq 0$ DISCRETE-TIME SYSTEM

CAUSAL ZERO INITIAL CONDITIONS

FIND $x[t]$, WHERE $x[t]$ IS CAUSAL $(x[t]=0,\ t<0)$

SUCH THAT $y[t] = h[t] * f[t]$ SATISFIES:

$$y[10] = 5$$

$$\sum_{j=0}^{10} y[j] = 2$$

AND SUCH THAT INPUT ENERGY $\sum_{t=0}^{10} |x[t]|^2$ IS MINIMIZED.

SINCE WE'RE LOOKING AT ENERGY FROM $t=0$ TO 10, LET'S DEFINE OUR INNER PRODUCT AS:

$$\langle x[t], y[t] \rangle = \sum_{t=0}^{10} x[t]\, y[t]$$

SO MINIMIZING ENERGY IS MINIMIZING THE NORM SQUARED:

$$\sum_{t=0}^{10} |x[t]|^2 = \langle x[t], x[t] \rangle = \|x[t]\|^2$$

NOW WE WANT TO PUT THE CONSTRAINTS INTO A FORM:

$$\langle x, y_1 \rangle = a_1$$

$$\langle x, y_2 \rangle = a_2$$

FIRST, WE RECOGNIZE THAT:

SINCE $h[t-n]=0$ FOR $n>t$

$$y[t] = x[t] * h[t] = \sum_{n=-\infty}^{\infty} x[n]h[t-n] = \sum_{n=0}^{t} x[n]h[t-n]$$

SINCE $x[n]=0$ FOR $n<0$

SO OUR FIRST CONSTRAINT IS:

$$y[10] = \sum_{n=0}^{10} x[n]h[10-n] = \langle x[n], \underbrace{h[10-n]}_{} \rangle = 5$$

$$y_1[n] = h[10-n] = (0.2)^{(10-n)} + 3(0.4)^{(10-n)}$$

AND OUR SECOND CONSTRAINT IS:

$$\underbrace{\sum_{j=0}^{10} y[j]}_{} = 2$$

SINCE $h[t]=0$ FOR $t<0$, $h[j-n]=0$ FOR $j<n\le 10$!

$$\sum_{j=0}^{10} \underbrace{\sum_{n=0}^{j} x[n]h[j-n]}_{y[j]} = \sum_{j=0}^{10}\sum_{n=0}^{10} x[n]h[j-n] = \sum_{n=0}^{10} x[n] \sum_{j=0}^{10} h[j-n]$$

LET $m = j-n$

$j = m+n$

$$= \sum_{n=0}^{10} x[n] \sum_{m+n=0}^{m+n=10} h[m] = \sum_{n=0}^{10} x[n] \sum_{m=-n}^{10-n} h[m]$$

$$= \sum_{n=0}^{10} x[n] \left(\sum_{m=0}^{10-n} h[m] \right) \Rightarrow y_2[n]$$

SINCE $h[m]=0$ FOR $m<0$.

$$= \langle x[n], \sum_{m=0}^{10-n} h[m] \rangle = 2$$

OR

$\langle x[n], y_2[n] \rangle = 2$ WHERE $y_2[n] = \sum_{m=0}^{10-n} (0.2)^m + 3(0.4)^m$

SO OUR SOLUTION WILL BE:

$$x[n] = c_1 y_1[n] + c_2 y_2[n]$$

WHERE c_1, c_2 SATISFY:

$$\begin{bmatrix} \langle y_1, y_1 \rangle & \langle y_2, y_1 \rangle \\ \langle y_1, y_2 \rangle & \langle y_2, y_2 \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

WITH: $y_1[n] = (0.2)^{10-n} + 3(0.4)^{10-n}$

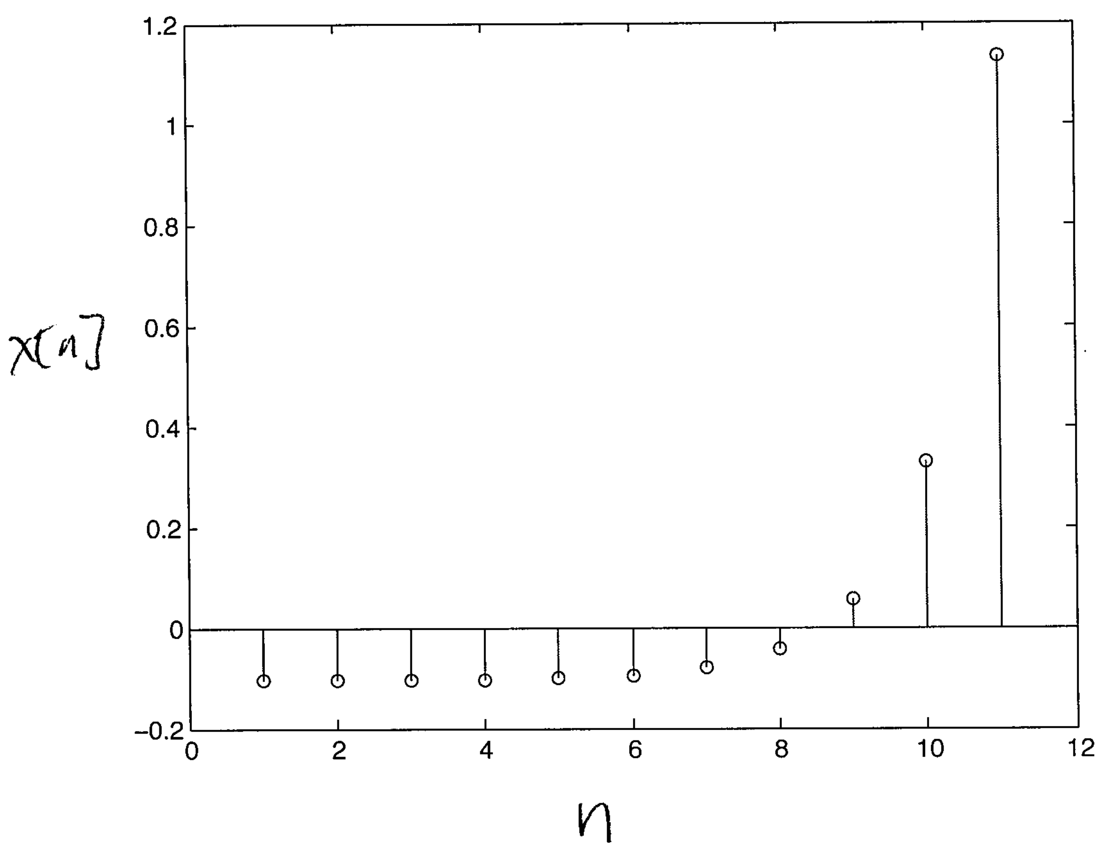
$$y_2[n] = \sum_{m=0}^{10-n} (0.2)^m + 3(0.4)^m$$

<u>PLUGGING INTO MATLAB:</u>

SEE NEXT PAGE

$$C = \begin{pmatrix} 0.300 \\ -0.017 \end{pmatrix}$$

$$x[n] = 0.3\, y_1[n] - 0.017\, y_2[n]$$



9

```
%
% Homework #4
%
% ECEn 671 - Fall 2010
%

% Problem 3-14.28
clear

t = linspace(0, 2, 10000);
dt = t(2) - t(1);

y1 = 3*exp(-2*(2-t)) + 4*exp(-5*(2-t));
y2 = 23/10 - 3/2*exp(-2*(2-t)) - 4/5*exp(-5*(2-t));

y1y1 = sum(y1.*y1)*dt;
y1y2 = sum(y1.*y2)*dt;
y2y2 = sum(y2.*y2)*dt;

A = [y1y1 y1y2; y1y2 y2y2];
a = [2; 3];

c = inv(A)*a;

disp(c);

plot(t, c(1)*y1 + c(2)*y2);


% Problem 3-14.29
clear

n = 0:10;

y1 = (0.2).^(10-n) + 3*(0.4).^(10-n);

z1 = (0.2).^n + 3*(0.4).^n;
for m = 0:10
    y2(m+1) = sum(z1(1:11-m));
end

y1y1 = sum(y1.*y1);
y1y2 = sum(y1.*y2);
y2y2 = sum(y2.*y2);

A = [y1y1 y1y2; y1y2 y2y2];
a = [5; 2];

c = inv(A)*a;

disp(c);

stem(c(1)*y1 + c(2)*y2);
```

MATLAB CODE

FOR

3-14.28

AND

3-14.29

3.15-30:

THIS IS AN UNDERCONSTRAINED SET OF EQUATIONS $A\underline{x} = \underline{b}$ SINCE A IS $m \times n$ w/ $m < n$.

HOWEVER, WE ARE MINIMIZING A WEIGHTED NORM SQUARED:

$$\|\underline{x}\|_Q \equiv \underline{x}^H Q \underline{x}$$, WHERE Q IS POSITIVE DEFINITE! AND SYMMETRIC!

WEIGHTED BY Q

LET'S RECAST THE PROBLEM INTO SOMETHING WE KNOW HOW TO SOLVE!

BOX 3.1 ON p. 134 GIVES US SOME PROPERTIES OF POSITIVE DEFINITE MATRICES. NOTICE THE 6^{TH}:

SINCE Q IS SYMMETRIC, IT IS REAL w/ $Q^T = Q$, SO IT IS ALSO HERMITIAN.

SO WE CAN WRITE (BY 6 ON p. 134):

$$Q = B^H B$$, WHERE B IS FULL RANK (AND THUS INVERTIBLE, SINCE IT IS SQUARE!)

NOW WE CAN REWRITE $\underline{x}^H Q \underline{x}$ AS:

$$\underline{x}^H Q \underline{x} = \underline{x}^H B^H B \underline{x} = (B\underline{x})^H B \underline{x} = \underline{z}^H \underline{z}$$ WHERE $\underline{z} = B\underline{x}$

PLUGGING IN $\underline{x} = B^{-1}\underline{z}$ TO $A\underline{x} = \underline{b}$, WE HAVE:

$$A\underline{x} = AB^{-1}\underline{z} = \underline{b}$$

SO WE HAVE RECAST THE PROBLEM AS:

MINIMIZE $\underline{z}^H \underline{z}$

SUBJECT TO $(AB^{-1})\underline{z} = \underline{b}$

WE KNOW HOW TO SOLVE THIS PROBLEM USING EQN. 3.91 ON p-183!

$$\underline{z} = (AB^{-1})^H \left(AB^{-1}(AB^{-1})^H\right)^{-1} \underline{b}$$

SO:

$$\underline{x} = B^{-1}\underline{z} = B^{-1}(AB^{-1})^H\left(AB^{-1}(AB^{-1})^H\right)^{-1}\underline{b}$$

$$\underline{x} = \underbrace{B^{-1}B^{-H}}A^H\left(A\underbrace{B^{-1}B^{-H}}A^H\right)^{-1}\underline{b}$$

$$\underline{x} = \underbrace{(B^HB)^{-1}}_{Q^{-1}}A^H\left(A(B^HB)^{-1}A^H\right)^{-1}\underline{b}$$

$$\boxed{\underline{x} = Q^{-1}A^H(AQ^{-1}A^H)^{-1}\underline{b}}$$

3.17-33:

$$g(t) = e^{-t/2} \quad \text{FOR} \quad 0 \le t \le \pi$$

$$f(t) = \sum_{k} g(t - k\pi)$$

a) FIND THE FOURIER SERIES COEFFICIENTS OF $f(t)$

$$a_k = \frac{1}{T}\int_0^{\pi} f(t)\, e^{-j\frac{2\pi}{T}kt}\, dt$$

$T = \pi$ AND $f(t) = g(t)$ ON INTERVAL $t = 0 \to \pi$, SO:

$$a_k = \frac{1}{\pi}\int_0^{\pi} e^{-t/2} e^{-j2kt}\, dt = \frac{1}{\pi}\int_0^{\pi} e^{-(\frac{1}{2}+j2k)t}\, dt$$

$$a_k = \frac{1}{\pi}\frac{-1}{(\frac{1}{2}+j2k)} e^{-(\frac{1}{2}+j2k)t}\Big/_0^{\pi} = \frac{1}{\pi}\frac{1 - e^{-(\frac{\pi}{2}+j2k\pi)}}{(\frac{1}{2}+j2k)} = \frac{1}{\pi}\frac{1 - e^{-\frac{\pi}{2}} e^{-jk2\pi}}{\frac{1}{2}+j2k}$$

just $= 1$!

$$\boxed{a_k = \frac{1}{\pi}\frac{(1 - e^{-\frac{\pi}{2}})}{(\frac{1}{2}+j2k)}} \quad \text{OR} \quad \boxed{a_k = \frac{2}{\pi}\frac{(1 - e^{-\frac{\pi}{2}})}{(1 + j4k)}}$$

b) FROM PARSEVAL, WE KNOW THAT:

$$\frac{1}{\pi}\int_0^{\pi} |f(t)|^2 dt = \sum_{k=-\infty}^{\infty}\left|\frac{2}{\pi}\frac{(1-e^{-\frac{\pi}{2}})}{(1+j4k)}\right|^2 = \sum_{k=-\infty}^{\infty}\frac{4}{\pi^2}\frac{(1-e^{-\frac{\pi}{2}})^2}{(1+j4k)(1-j4k)}$$

REMEMBER: $|c|^2 = \bar{c}c$!

$$\frac{1}{\pi}\int_0^{\pi}(e^{-t/2})^2 dt = \sum_{k=-\infty}^{\infty}\frac{4}{\pi^2}\frac{(1-e^{-\frac{\pi}{2}})^2}{(1+16k^2)} = (1-e^{-\frac{\pi}{2}})^2 \underbrace{\sum_{k=-\infty}^{\infty}\left(\frac{4}{\pi^2}\frac{1}{1+16k^2}\right)}_{\text{THIS IS THE SUM WE WANT!}}$$

SO:

$$\sum_{k=-\infty}^{\infty}\frac{4}{\pi^2}\frac{1}{1+16k^2} = \frac{1}{\pi(1-e^{-\frac{\pi}{2}})^2}\int_0^{\pi} e^{-t}\, dt = \boxed{\frac{1-e^{-\pi}}{\pi(1-e^{-\frac{\pi}{2}})^2}}$$

3.18-34:

SHOW THAT:

$$T_n(t) = \cos(n\cos^{-1}t)$$ ← THE CHEBYSHEV POLYNOMIALS

SATISFY THE RECURRENCE:

$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t) \quad , \quad T_0(t) = 1, \quad T_1(t) = t$$

START WITH:

$$T_{n+1}(t) = \cos((n+1)\cos^{-1}t) = Re\left\{e^{j(n+1)\cos^{-1}t}\right\}$$

$$= Re\left\{e^{jn\cos^{-1}t}e^{j\cos^{-1}t}\right\}$$

$$= Re\left\{\left[\cos(n\cos^{-1}t) + j\sin(n\cos^{-1}t)\right]\left[\underbrace{\cos(\cos^{-1}t)}_{t} + j\sin(\cos^{-1}t)\right]\right\}$$

$$= Re\left\{t\cos(n\cos^{-1}t) + j(\text{blah}) + j(\text{blah}) - \sin(n\cos^{-1}t)\sin(\cos^{-1}t)\right\}$$

$$= t\cos(n\cos^{-1}t) - \sin(n\cos^{-1}t)\sin(\cos^{-1}t)$$

$$= t\cos(n\cos^{-1}t) - \frac{\left(e^{jn\cos^{-1}t} - e^{-jn\cos^{-1}t}\right)}{2j}\frac{\left(e^{j\cos^{-1}t} - e^{-j\cos^{-1}t}\right)}{2j}$$

$$= t\cos(n\cos^{-1}t) + \frac{1}{4}\left(e^{j(n+1)\cos^{-1}t} - e^{j(n-1)\cos^{-1}t} - e^{-j(n-1)\cos^{-1}t} + e^{-j(n+1)\cos^{-1}t}\right)$$

$$= t\cos(n\cos^{-1}t) + \frac{1}{4}\left(\underbrace{e^{j(n+1)\cos^{-1}t} + e^{-j(n+1)\cos^{-1}t}}_{2\cos((n+1)\cos^{-1}t)} \underbrace{- e^{j(n-1)\cos^{-1}t} - e^{-j(n-1)\cos^{-1}t}}_{-2\cos((n-1)\cos^{-1}t)}\right)$$

$$T_{n+1}(t) = \underbrace{t\cos(n\cos^{-1}t)}_{T_n(t)} + \frac{1}{2}\underbrace{\cos((n+1)\cos^{-1}t)}_{T_{n+1}(t)} - \frac{1}{2}\underbrace{\cos((n-1)\cos^{-1}t)}_{T_{n-1}(t)}$$

$$T_{n+1}(t) = tT_n(t) + \frac{1}{2}T_{n+1}(t) - \frac{1}{2}T_{n-1}(t)$$

$$\frac{1}{2}T_{n+1}(t) = tT_n(t) - \frac{1}{2}T_{n-1}(t)$$

$$\boxed{T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t)}$$ ✓

FOR $|t|>1$, WE HAVE:

$$T_n(t) = \cosh(n\cosh^{-1}t)$$

$$\cosh(t) = \frac{e^t + e^{-t}}{2} = \cosh(-t)$$

$$e^t = \cosh t + \sinh(t)$$

SO:

$$T_{n+1}(t) = \cosh\left((n+1)\cosh^{-1}t\right) = \frac{1}{2}\left(e^{(n+1)\cosh^{-1}t} + e^{-(n+1)\cosh^{-1}t}\right)$$

$$= \frac{1}{2}\left(e^{n\cosh^{-1}t}e^{\cosh^{-1}t} + e^{-n\cosh^{-1}t}e^{-\cosh^{-1}t}\right)$$

$$= \frac{1}{2}\Big[\big[\underbrace{\cosh(n\cosh^{-1}t)}_{T_n(t)} + \sinh(n\cosh^{-1}t)\big]\big[\underbrace{\cosh(\cosh^{-1}t)}_{t} + \sinh(\cosh^{-1}t)\big]$$
$$+ \big[\underbrace{\cosh(-n\cosh^{-1}t)}_{T_n(t)} + \sinh(-n\cosh^{-1}t)\big]\big[\underbrace{\cosh(-\cosh^{-1}t)}_{t} + \sinh(-\cosh^{-1}t)\big]\Big]$$

$$= \frac{1}{2}\Big[\big(T_n(t) + \sinh(n\cosh^{-1}t)\big)\big(t + \sinh(\cosh^{-1}t)\big)$$
$$+ \big(T_n(t) - \sinh(n\cosh^{-1}t)\big)\big(t - \sinh(\cosh^{-1}t)\big)\Big]$$

$$= \frac{1}{2}\Big[t\,T_n(t) + \cancel{\sinh(\cosh^{-1}t)\,T_n(t)} + \cancel{t\sinh(n\cosh^{-1}t)} + \sinh(n\cosh^{-1}t)\sinh(\cosh^{-1}t)$$
$$+ t\,T_n(t) - \cancel{\sinh(\cosh^{-1}t)\,T_n(t)} - \cancel{t\sinh(n\cosh^{-1}t)} + \sinh(n\cosh^{-1}t)\sinh(\cosh^{-1}t)\Big]$$

$$T_{n+1}(t) = t\,T_n(t) + \sinh(n\cosh^{-1}t)\sinh(\cosh^{-1}t)$$

$$T_{n+1}(t) = t\,T_n(t) + \frac{\left(e^{n\cosh^{-1}t} - e^{-n\cosh^{-1}t}\right)}{2}\frac{\left(e^{\cosh^{-1}t} - e^{-\cosh^{-1}t}\right)}{2}$$

$$T_{n+1}(t) = t\,T_n(t) + \frac{1}{4}\left(e^{(n+1)\cosh^{-1}t} - e^{(n-1)\cosh^{-1}t} - e^{-(n-1)\cosh^{-1}t} + e^{-(n+1)\cosh^{-1}t}\right)$$

$$T_{n+1}(t) = t\,T_n(t) + \frac{1}{4}\left(2\underbrace{\cosh((n+1)\cosh^{-1}t)}_{T_{n+1}(t)} - 2\underbrace{\cosh((n-1)\cosh^{-1}t)}_{T_{n-1}(t)}\right)$$

$$T_{n+1}(t) = t\,T_n(t) + \frac{1}{2}T_{n+1}(t) - \frac{1}{2}T_{n-1}(t)$$

$$\frac{1}{2}T_{n+1}(t) = t\,T_n(t) - \frac{1}{2}T_{n-1}(t)$$

$$\boxed{T_{n+1}(t) = 2t\,T_n(t) - T_{n-1}(t)}$$ ✓

3.18-37:

a) $\int_{x=a}^{x=b} g(x)\,dx$ LET: $t = \frac{1}{b-a}(2x-a-b) \longrightarrow (b-a)t = 2x-a-b$

$2x = (b-a)t + a + b$

$x = \frac{(b-a)t+a+b}{2}$

$dt = \frac{2}{b-a}\,dx$

$dx = \frac{b-a}{2}\,dt$

SUBSTITUTING FOR x:

$\frac{(b-a)t+a+b}{2} = b$ ← SOLVE FOR $t \Rightarrow$

$(b-a)t+a+b = 2b$

$(b-a)t = 2b-b-a$

$(b-a)t = b-a$

$\boxed{t=1}$

$$\int g\left(\frac{(b-a)t+a+b}{2}\right)\frac{b-a}{2}\,dt$$

$\underbrace{\frac{(b-a)t+a+b}{2} = a}$

SOLVE FOR $t \Rightarrow (b-a)t+a+b = 2a$

$(b-a)t = 2a-a-b$

$(b-a)t = a-b$

$t = \frac{a-b}{b-a} = -1$

$\boxed{t=-1}$

SO: $\boxed{\int_a^b g(x)\,dx = \int_{-1}^{1} f(t)\,dt \quad \text{WHERE: } f(t) = \frac{b-a}{2}\, g\left(\frac{(b-a)t+a+b}{2}\right)}$ ✓

b) IF THE $P_n(t)$ FOR $n=0,1,\dots,m$ ARE ORTHOGONAL OVER $[-1,1]$, THEN THEY ARE LINEARLY INDEPENDENT. THE SPACE OF POLYNOMIALS OF DEGREE $\leq m$ IS $(m+1)$-DIMENSIONAL, SO THE m ORTHOGONAL POLYNOMIALS $\{P_n(t), n=0,1,\dots,m-1\}$ MUST SPAN THE SPACE OF ALL POLYNOMIALS OF DEGREE $\leq m-1$.

THUS, WE CAN WRITE $p(t)$ AS A LINEAR COMBINATION OF THE FIRST m POLYNOMIALS $P_n(t)$:

SINCE $p(t)$ IS OF DEGREE $\leq m-1$

$$p(t) = \sum_{n=0}^{m-1} a_n P_n(t)$$

SO:

$$\langle p(t), P_m(t)\rangle = \left\langle \sum_{n=0}^{m-1} a_n P_n(t), P_m(t)\right\rangle$$

$$= \sum_{n=0}^{m-1} a_n \langle P_n(t), P_m(t)\rangle = 0$$

SINCE $n \neq m$ FOR THE ENTIRE SUMMATION AND THE $P_n(t)$ ARE MUTUALLY ORTHOGONAL.

c) SINCE $f(t)$ IS OF DEGREE $2m-1$, IT IS IN THE SPACE SPANNED BY $\{P_n(t), n=0,1,\dots,2m-1\}$

AND WE CAN WRITE:

$$f(t) = \sum_{k=0}^{2m-1} c_k P_k(t) = \sum_{k=m}^{2m-1} c_k P_k(t) + \sum_{k=0}^{m-1} c_k P_k(t)$$

$$f(t) = P_m(t) \underbrace{\sum_{k=m}^{2m-1} c_k \frac{P_k(t)}{P_m(t)}}_{\text{POLYNOMIAL OF DEGREE} \leq m-1} + \underbrace{\sum_{k=0}^{m-1} c_k P_k(t)}_{\text{POLYNOMIAL OF degree} \leq m-1}$$

NOTICE THAT $\sum_{k=m}^{2m-1} c_k \frac{P_k(t)}{P_m(t)}$ IS A POLYNOMIAL

OF DEGREE $\leq m-1$, SINCE k RUNS FROM

$k = m \to 2m-1$ AND THE DEGREE OF $\frac{P_k(t)}{P_m(t)}$ IS

$k-m$, WHICH MEANS

$\frac{P_k(t)}{P_m(t)}$ IS OF DEGREE 0 (FOR $k=m$)

UP TO DEGREE $m-1$ (FOR $k=2m-1$).

SO DEFINING: $q(t) = \sum_{k=m}^{2m-1} c_k \frac{P_k(t)}{P_m(t)}$

AND $r(t) = \sum_{k=0}^{m-1} c_k P_k(t)$

WE HAVE:

$$\boxed{f(t) = q(t) P_m(t) + r(t)}$$ ✓

d) $q(t)$ AND $r(t)$ ARE OF DEGREE $\leq m-1$, SO THEY ARE SPANNED BY $\{P_n(t),\ n=0,1,\dots,m-1\}$ AND WE CAN WRITE:

$$\boxed{q(t) = \sum_{k=0}^{m-1} \alpha_k P_k(t) \quad \text{AND} \quad r(t) = \sum_{k=0}^{m-1} \beta_k P_k(t)}$$

e) $$\int_{-1}^{1} f(t)\,dt = \int_{-1}^{1}\big(q(t)\,P_m(t) + r(t)\big)\,dt = \int_{-1}^{1}\sum_{k=0}^{m-1}\alpha_k P_k(t)P_m(t)\,dt + \int_{-1}^{1}\sum_{k=0}^{\infty}\beta_k P_k(t)\,dt$$

$$= \sum_{k=0}^{\infty}\alpha_k \underbrace{\cancelto{0}{\int_{-1}^{1} P_k(t)P_m(t)\,dt}} + \underbrace{\sum_{k=0}^{\infty}\beta_k\int_{-1}^{1} P_k(t)\,dt}$$

SINCE $k \neq m$ ACROSS THE SUM, AND $P_k(t)$ AND $P_m(t)$ ARE ORTHOGONAL FOR $k \neq m$!

THIS IS THE TRICK!

$$= \sum_{k=0}^{\infty}\beta_k\int_{-1}^{1} P_k(t)\,dt = \sum_{k=0}^{\infty}\beta_k\int_{-1}^{1} P_k(t)\,\frac{P_0(t)}{P_0(t)}\,dt$$

BUT $P_0(t)$ IS A CONSTANT, SINCE IT IS A ZERO-DEGREE POLYNOMIAL!

WE CAN PULL IT OUT OF THE INTEGRAL!

$$= \sum_{k=0}^{\infty}\frac{\beta_k}{P_0(t)}\underbrace{\int_{-1}^{1} P_k(t)\,P_0(t)\,dt}$$

BY ORTHOGONALITY, THIS IS ZERO FOR $k \neq 0$!

$$= \frac{\beta_0}{P_0(t)}\int_{-1}^{1}\big(P_0(t)\big)^2\,dt = \boxed{\beta_0\int_{-1}^{1} P_0(t)\,dt}$$

f) $$\sum_{i=1}^{m} a_i f(t_i) = \sum_{i=1}^{m} a_i \Big[g(t_i) \underbrace{\cancel{p_m(t_i)}}_{\to 0} + r(t_i) \Big]$$

→ 0 SINCE t_i ARE ROOTS OF $p_m(t)$!

$$= \sum_{i=1}^{m} a_i r(t_i) = \sum_{i=1}^{m} a_i \sum_{k=0}^{m-1} \beta_k p_k(t_i)$$

$$= \boxed{\sum_{k=0}^{m-1} \beta_k \sum_{i=1}^{m} a_i p_k(t_i)}$$

g) WE CAN WRITE THE WEIGHT CONSTRAINTS AS:

$$\sum_{i=1}^{m} a_i p_k(t_i) = \delta[k] \int_{-1}^{1} p_0(t)\,dt$$

ONLY NONZERO FOR $K=0$.

THEN:

$$\sum_{i=1}^{m} a_i f(t_i) = \sum_{k=0}^{m-1} \beta_k \underbrace{\sum_{i=1}^{m} a_i p_k(t_i)}_{\delta[k] \int_{-1}^{1} p_0(t)\,dt} = \underbrace{\sum_{k=0}^{m-1} \beta_k \, \delta[k] \int_{-1}^{1} p_0(t)\,dt}$$

ONLY $K=0$ TERM SURVIVES.

$$= \boxed{\beta_0 \int_{-1}^{1} p_0(t)\,dt}$$

h) FROM PART (g) WE HAD:

$$\sum_{i=1}^{m} a_i P_k(t_i) = \begin{cases} \int_{-1}^{1} P_0(t)\,dt & , \ k=0 \\ 0 & , \ k=1,2,\dots,m-1 \end{cases}$$

THIS IS A SET OF m EQUATIONS FOR $k=0,\dots,m-1$:

$\underline{k=0:}$ $$\sum_{i=1}^{m} a_i P_0(t_i) = \int_{-1}^{1} P_0(t)\,dt$$

$\underline{k=1:}$ $$\sum_{i=1}^{m} a_i P_1(t_i) = 0$$

$$\vdots$$

$\underline{k=m-1:}$ $$\sum_{i=1}^{m} a_i P_{n-1}(t_i) = 0$$

AND CAN BE WRITTEN AS:

$$\boxed{\begin{bmatrix} P_0(t_1) & P_0(t_2) & \cdots & P_0(t_m) \\ P_1(t_1) & P_1(t_2) & \cdots & P_1(t_m) \\ \vdots & \vdots & & \vdots \\ P_{m-1}(t_1) & P_{m-1}(t_2) & \cdots & P_{m-1}(t_m) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} \int_{-1}^{1} P_0(t)\,dt \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$$

i) FROM (e) $\int_{-1}^{1} f(t)\,dt = \beta_0 \int_{-1}^{1} P_0(t)\,dt$

AND from (g) $\sum_{i=1}^{m} a_i f(t_i) = \beta_0 \int_{-1}^{1} P_0(t)\,dt$

SO:

$$\boxed{\int_{-1}^{1} f(t)\,dt = \sum_{i=1}^{m} a_i f(t_i)}$$

j) FOR A WEIGHTED INNER PRODUCT, WE'LL NEED TO MAKE SOME CHANGES STARTING AT STEP (e):

$$(e) \quad \int_{-1}^{1} w(t) f(t)\,dt = \int_{-1}^{1} \big(q(t)P_m(t) + r(t)\big) w(t)\,dt$$

$$= \cancelto{0}{\int_{-1}^{1} q(t)P_m(t) w(t)\,dt} + \int_{-1}^{1} r(t) w(t)\,dt$$

$$= \sum_{k=0}^{m-1} \beta_k \int_{-1}^{1} P_k(t)\,w(t)\,dt = \sum_{k=0}^{m-1} \frac{\beta_k}{P_0(t)} \int_{-1}^{1} P_k(t)P_0(t)\,w(t)\,dt$$

$$= \beta_0 \int_{-1}^{1} P_0(t)\,w(t)\,dt$$

IN PART (g), WE CHOOSE:

$$(g) \quad \sum_{i=1}^{m} a_k P_k(t_i) = \delta[k] \int_{-1}^{1} P_0(t)\,w(t)\,dt$$

NEED THIS!

SO WE HAVE:

$$\sum_{i=1}^{m} a_i f(t_i) = \beta_0 \int_{-1}^{1} P_0(t)\,w(t)\,dt$$

NOW EQUATING RESULTS FROM (e) AND (g) WE HAVE:

$$\int_{-1}^{1} f(t)\,w(t)\,dt = \sum_{i=1}^{m} a_i f(t_i)$$

WHERE OUR a_i ARE NOW GIVEN BY:

$$\begin{bmatrix} P_0(t_1) & \cdots & P_0(t_m) \\ \vdots & & \vdots \\ P_{m-1}(t_1) & \cdots & P_{m-1}(t_m) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} \int_{-1}^{1} P_0(t)\,w(t)\,dt \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

AN INTERESTING OBSERVATION: THE a_i CAN BE PRE-COMPUTED FOR ANY SET OF ORTHOGONAL POLYNOMIALS $\{P_k(t)\}$, MAKING EVALUATION OF INTEGRALS USING GAUSSIAN QUADRATURE VERY COMPUTATIONALLY EFFICIENT.