4.1-1:

$$\dddot{x}(t) - 2\ddot{x}(t) - \dot{x}(t) = b(t)$$

$$\frac{d^3}{dt^3}x(t) - 2\frac{d^2}{dt^2}x(t) - \frac{d}{dt}x(t) = b(t)$$

$$\boxed{L = \frac{d^3}{dt^3} - 2\frac{d^2}{dt^2} - \frac{d}{dt} \qquad Lx(t) = b(t)}$$

4.2-4:

$$A_1 = \begin{bmatrix} 4 & 3 \\ 3 & 6 \end{bmatrix}$$

ℓ_1 NORM = LARGEST COLUMN SUM = $\boxed{9} = \|A_1\|_1$

ℓ_2 NORM = $\sqrt{\max_i |\lambda_i|}$ WHERE λ_i ARE EIGENVALUES OF $A_1^H A_1$.

$$A_1^H A_1 = \begin{bmatrix} 4 & 3 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 25 & 30 \\ 30 & 45 \end{bmatrix}$$

CHARACTERISTIC EQN. OF $A_1^H A_1$ IS:

$$\det\begin{bmatrix} 25-\lambda & 30 \\ 30 & 45-\lambda \end{bmatrix} = 0$$

$$(25-\lambda)(45-\lambda) - 900 = 0$$

$$1125 - 70\lambda + \lambda^2 - 900 = 0$$

$$\lambda^2 - 70\lambda + 225 = 0$$

$\lambda_1 = 66.6228$

$\lambda_2 = 3.3772$

SO:

$\|A_1\|_2 = \boxed{8.1623}$

$\|A_1\|_\infty$ = LARGEST ROW SUM

$\|A_1\|_F = \boxed{\sqrt{70}}$

$\boxed{\|A_1\|_\infty = 9}$

$A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ $\boxed{\|A_2\|_1 = 4}$

$A_2^H A_2 = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 2 & 4 \end{bmatrix}$

$\det \begin{vmatrix} 10-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$ $(10-\lambda)(4-\lambda) - 4 = 0$ $\lambda_1 = 10.6$

$40 - 14\lambda + \lambda^2 - 4 = 0$ $\lambda_2 = 3.39$

$\lambda^2 - 14\lambda + 36 = 0$

$\lambda = \dfrac{14 \pm \sqrt{14^2 - 4 \cdot 36}}{2}$ $\max_i |\lambda_i| = \sqrt{10.6}$

so:

$\boxed{\|A_2\|_2 = 3.2566}$

$\|A_2\|_F = \sqrt{1+4+9+0} = \boxed{\sqrt{14} = \|A_2\|_F}$

$\boxed{\|A_2\|_\infty = 3}$

$A_3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\boxed{\|A_3\|_1 = 3}$

$A_3^H A_3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ $\det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 5-\lambda \end{bmatrix} = 0$

$(1-\lambda)(5-\lambda) - 4 = 0$ $\lambda_1 = 0.1716$

$5 - 6\lambda + \lambda^2 - 4 = 0$ $\lambda_2 = 5.8284$

$\lambda^2 - 6\lambda + 1 = 0$

$\boxed{\|A_3\|_2 = 2.4142}$ $\boxed{\|A_3\|_F = \sqrt{6}}$ $\boxed{\|A_3\|_\infty = 3}$

4.2-10:

SINCE F IS AN OPERATOR AND $\|\cdot\|$ SATISFIES MULTIPLICATIVE PROPERTY AND $\|F\|<1$, BY THEOREM 4.3 WE HAVE:

$$(I-F)^{-1}=\sum_{i=0}^{\infty}F^{i}$$

TAKING NORM OF BOTH SIDES:

$$\|(I-F)^{-1}\|=\Big\|\sum_{i=0}^{\infty}F^{i}\Big\|\le\sum_{i=0}^{\infty}\|F^{i}\|$$

$\Uparrow$ TRIANGLE INEQUALITY FOR NORMS

$$\sum_{i=0}^{\infty}\|F^{i}\|=1+\|F\|+\|F^2\|+\dots\le 1+\|F\|+\|F\|^2+\|F\|^3+\dots$$

$\Uparrow$ BY THE SUBMULTIPLICATIVE PROPERTY APPLIED TO EACH TERM:

$\|F^2\|\le\|F\|^2$

$\|F^3\|\le\|F\|^3$

$\vdots$

etc.

BUT WE HAVE:

$$1+\|F\|+\|F\|^2+\|F\|^3+\dots=\frac{1}{1-\|F\|}$$

SINCE $\|F\|<1$.

SO:

$$\boxed{\|(I-F)^{-1}\|\le\frac{1}{1-\|F\|}}$$

✓

4.2-15:

LET $B = A^H A$.

$$\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & & \vdots \\ \vdots & & & \vdots \\ A_{1n} & \cdots & & A_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & & \vdots \\ \vdots & & & \vdots \\ A_{n1} & \cdots & & A_{nn} \end{bmatrix} = \begin{bmatrix} B_{11} & \cdots & & B_{1n} \\ B_{21} & B_{22} & & \vdots \\ \vdots & & & \vdots \\ B_{n1} & \cdots & & B_{nn} \end{bmatrix}$$

$$B_{ij} = \sum_{k=1}^{n} \overline{A_{ki}} A_{kj}$$

$$tr(A^H A) = tr(B) = \sum_{m=1}^{n} B_{mm} = \sum_{m=1}^{n} \sum_{k=1}^{n} \overline{A_{km}} A_{km}$$

$$= \sum_{m=1}^{n} \sum_{k=1}^{n} |A_{km}|^2 = \|A\|_F^2$$ BY DEFINITION OF THE FROBENIUS NORM. ✓

4.3-25:

IF THE INVERSE A^{-1} EXISTS, WE HAVE:

$$AA^{-1} = I \quad \text{AND} \quad A^{-1}A = I$$

NOW, NOTICE THAT $I^* = I$. ⇐ PLUG IT IN TO DEF. OF ADJOINT TO CONVINCE YOURSELF!

SO:

$$I^* = (AA^{-1})^* = I$$

$$\boxed{(A^{-1})^* A^* = I}$$

AND:

$$I^* = (A^{-1}A)^* = I$$

$$\boxed{A^*(A^{-1})^* = I}$$

SO: $(A^{-1})^* A^* = A^*(A^{-1})^* = I$

WHICH MEANS: $(A^{-1})^* = (A^*)^{-1}$ ✓.

4.5-29:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \\ 3 & 12 \end{bmatrix} \qquad \boxed{\mathcal{R}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}}$$

$$A^* = A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 12 \end{bmatrix} \qquad \boxed{\mathcal{R}(A^*) = \text{span}\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}}$$

$A\underline{x} = \underline{0}$

$$\begin{bmatrix} 1 & 4 \\ 2 & 8 \\ 3 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} x_1 + 4x_2 &= 0 \\ x_1 &= -4x_2 \end{aligned}$$

SO: $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$ IS IN $\mathcal{N}(A)$

$$\boxed{\mathcal{N}(A) = \text{span}\left\{ \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}}$$

$A^T y = \underline{0}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 12 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad y_1 + 2y_2 + 3y_3 = 0$$

LET $y_3 = 0$.

THEN: $y_1 = -2y_2 \Rightarrow \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

LET $y_2 = 0$:

THEN: $y_1 = -3y_3 \qquad \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

SO: $\boxed{\mathcal{N}(A^*) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}}$