Lecture 1

There are three common mathematical tools for representing dynamic systems.

(Def: <u>dynamic system</u> - a system that moves, or has memory, or consumes or generates energy.
- For mechanical systems - anything with positions, velocities, angles, etc.
- For computer systems - anything with memory.

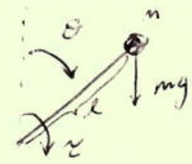
Methods for representing dynamic systems.

- Differential equation (ordinary and partial)
- transfer functions (limited to LTI - lumped element)
- state space (limited to lumped element)

State space models are by far the most useful
- apply to largest class of problems
- easy to account for complex inter-connections
- Standard computation methods for solving
- powerful mathematical tools for analysis and design.

Examples

Pendulum



Newtons law $\quad ml\ddot{\theta} = \underbrace{mg\sin\theta}_{\text{gravity}} - \underbrace{kl\dot{\theta}}_{\text{friction}} + \underbrace{\tau}_{\text{external torque}}$ (3.1)

Eq (3.1) is an ODE representation of the system.

We can create several different state space representations:

Select

$$x_1 \triangleq \theta \qquad y \triangleq \theta$$
$$x_2 \triangleq \dot{\theta}$$
$$u \triangleq \tau$$

then
$$\dot{x}_1 = \dot{\theta} = x_2$$
$$\dot{x}_2 = \ddot{\theta} = mg\sin\theta - kl\dot{\theta} + \tau = mg\sin x_1 - kl\cdot x_2 + u$$

∴ The nonlinear state space model is

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix} x_2 \\ mg\sin x_1 - kl x_2 + u \end{pmatrix}}_{f(x,u)}$$

$$y = \underbrace{x_1}_{h(x,u)}$$

Using the linear approximation $\sin x_1 \approx x_1$ gives

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix} 0 & 1 \\ mg & -kl \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} \underbrace{u}_{u}$$

$$y = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{} + \underbrace{0}_{D} \cdot \underbrace{u}_{u}$$

Alternatively we could have chosen

$$x_1 = \theta + \dot\theta \quad\Rightarrow\quad \theta = x_1 - \dot\theta = x_1 - 2x_2$$

$$x_2 = \tfrac{1}{2}\dot\theta \quad\Rightarrow\quad \dot\theta = 2x_2$$

then

$$y = \theta$$

$$u = \tau$$

Then

$$\dot x_1 = \dot\theta + \ddot\theta = 2x_2 + mg\theta - kl\dot\theta + \tau$$

$$= 2(1-kl)x_2 + mg(x_1 - 2x_2) + u$$

$$= mg\,x_1 + 2(1-kl-mg)x_2 + u$$

$$\dot x_2 = \tfrac{1}{2}\ddot\theta = \tfrac{1}{2}(mg\theta - kl\dot\theta + \tau)$$

$$= \tfrac{1}{2}(mg(x_1 - 2x_2) - 2kl\,x_2 + u)$$

$$= \tfrac{1}{2}mg\,x_1 - \tfrac{1}{2}(mg + 2kl)x_2 + \tfrac{1}{2}u$$

$$\therefore \quad \begin{pmatrix} \dot x_1 \\ \dot x_2 \end{pmatrix} = \begin{pmatrix} mg & 2(1-kl-mg) \\ \frac{1}{2}mg & -\frac{1}{2}(mg+2kl) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0\,u$$

Different state space model, but same system!

The transfer function model is

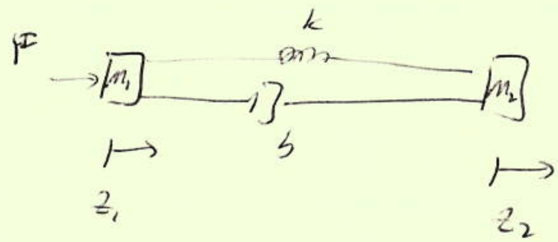
$$ml\,s^2\Theta(s) - mg\,\Theta(s) + kl\,s\,\Theta(s) = \tau(s)$$

$$\Rightarrow ml\left(s^2 + \frac{k}{m}s - g/l\right)\Theta(s) = \tau(s)$$

$$\Rightarrow \quad \Theta(s) = \left(\frac{1/ml}{s^2 + \frac{k}{m}s - g/l}\right)\tau(s)$$

Example

Coupled Mass spring damper



$$m_1 \ddot{z}_1 = k(z_2 - z_1) + b(\dot{z}_2 - \dot{z}_1) + F$$
$$m_2 \ddot{z}_2 = k(z_1 - z_2) + b(\dot{z}_1 - z_2)$$
} differential equation

— coupled 2^{nd} order system

Let

$x_1 = z_1$ $\qquad y_1 = z_1$

$x_2 = \dot{z}_1$ $\qquad y_2 = z_2$

$x_3 = z_2$

$x_4 = \dot{z}_2$ $\qquad u = F$

then

$$\dot{x}_1 = \dot{z}_1 = x_2$$
$$\dot{x}_2 = \ddot{z}_1 = \frac{k}{m_1}(z_2 - z_1) + \frac{b}{m_1}(\dot{z}_2 - \dot{z}_1) + \frac{1}{m_1}F = \frac{k}{m_1}(x_3 - x_1) + \frac{b}{m_1}(x_4 - x_2) + \frac{1}{m_1}u$$
$$\dot{x}_3 = \dot{z}_2 = x_4$$
$$\dot{x}_4 = \ddot{z}_2 = \frac{k}{m_2}(x_1 - x_3) + \frac{b}{m_2}(x_2 - x_4)$$

The state space model is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & -\frac{b}{m_1} & \frac{k}{m_1} & \frac{b}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & \frac{b}{m_2} & -\frac{k}{m_2} & -\frac{b}{m_2} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1/m_1 \\ 0 \\ 0 \end{pmatrix}}_{B} u$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{D} u$$

Discrete time, linear state space model

$$x(t+1) = A(t)\,x(t) + B(t)\,u(t)$$

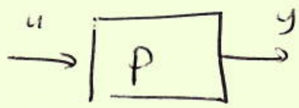
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

or for LTI systems

$$x^+ = Ax + Bu$$

$$y = Cx + Du$$

Block Diagrams



We are familiar with the representation using transfer functions

$$P: \quad G(s)$$

In this ~~both/later we~~ class we focus on state-space representation

$$P: \quad \dot{x} = Ax + Bu, \quad y = Cx + Du$$

Lets look at composition rules for state space:

Let $\quad P_1: \; \dot{x}_1 = A_1 x_1 + B_1 u_1, \quad y_1 = C_1 x_1 + D_1 u_1$

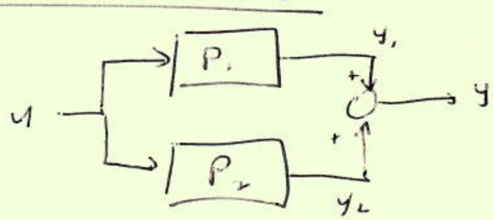
$\quad P_2: \; \dot{x}_2 = A_2 x_2 + B_2 u_2, \quad y_2 = C_2 x_2 + D_2 u_2$

or ~~in~~ in transfer function format

$P_1: \; G_1(s)$

$P_2: \; G_2(s)$

Parallel Interconnection



Transfer function

$$y = y_1 + y_2$$
$$= G_1(s)\,U(s) + G_2(s)\,U(s)$$
$$\Rightarrow \; Y(s) = \left(G_1(s) + G_2(s)\right) U(s)$$

State space

~~state~~ $\dot{x}_1 = Ax_1 + B_1 u, \quad y_1 = C_1 x_1 + D_1 u$

$\dot{x}_2 = A_2 x_2 + B_2 u, \quad y_2 = C_2 x_2 + D_2 u$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u$$

$$y = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (D_1 + D_2)\, u$$

Cascade Interconnection

$$\xrightarrow{u} \boxed{P_1} \xrightarrow{z} \boxed{P_2} \xrightarrow{y}$$

Transfer funct.

$$y = G_2(s)\, z$$

$$z = \cancel{P} G_1(s)\, u$$

$$\Rightarrow \quad y = G_2(s)\, G_1(s)\, u$$

state space

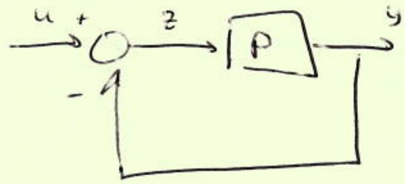
$$\dot{x}_1 = A_1 x_1 + B_1 u \,, \quad z = C_1 x_1 + D_1 u$$

$$\dot{x}_2 = A_2 x_2 + B_2 z \,, \quad y = C_2 x_2 + D_2 z$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 D_1 \end{pmatrix} u$$

$$y = \begin{pmatrix} D_2 C_1 & \vdots & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + D_2 D_1 u$$

Feedback Interconnection



transfer function

$$Y(s) = G_0(s)\left(U(s) - Y(s)\right)$$

$$\Rightarrow \left(I + G(s)\right)Y(s) = G(s)\,U(s)$$

$$\Rightarrow \quad Y(s) = \left(I + G(s)\right)^{-1} G(s)\,U(s)$$

State space

$$\dot{x} = Ax + Bz$$

$$y = Cx + Dz$$

$$z = u - y$$

$$= u - Cx - Dz$$

$$\Rightarrow (I + D)z = u - Cx$$

$$\Rightarrow \quad z = (I+D)^{-1}u - (I+D)^{-1}Cx$$

So

$$\dot{x} = \left(A - B(I+D)^{-1}C\right)x + B(I+D)^{-1}u$$

$$y = \left(C - D(I+D)^{-1}C\right)x + D(I+D)^{-1}u$$