Lecture 12 - Controllable Systems

12.1 Controllable Systems

Consider the continuous / discrete-time system

$$\dot{x} = Ax + Bu \quad , \quad x^+ = Ax + Bu \tag{1.1}$$

<u>Def</u>: the system (1.1) (continuous or discrete) or the pair (A, B) is said to be <u>reachable on $[t_0, t_1]$</u> if $\mathcal{R}[t_0, t_1] = \mathbb{R}^n$, ie the origin can be transferred to any state

<u>Def</u> The system (1.1.) or the pair (A, B) is said to be controllable on $[t_0, t_1]$ if $\mathcal{C}[t_0, t_1] = \mathbb{R}$, ie every state can be transferred to the origin.

<u>Thm 12.1</u> The LTI system (1.1) is controllable iff

$$\text{rank } \mathcal{C} = n$$

where $\mathcal{C} = [B \ AB \ \cdots \ A^{n-1}B]$

12. Eigenvector test for controllability

Def Given an $n \times n$ matrix A, a linear subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is said to be <u>A-invariant</u> if for every $v \in \mathcal{V}$ we have $Av \in \mathcal{V}$

Example For the matrix $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ where

$$A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a v_1 \\ 0 \end{pmatrix}$$

we have that $\mathcal{V} = \left\{ x = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \alpha \in \mathbb{R} \right\}$ is A-invariant

since $v \in \mathcal{V} \Rightarrow \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \begin{pmatrix} a\alpha \\ 0 \end{pmatrix} \in \mathcal{V}$

Properties Given an $n \times n$ matrix A and a non-zero A-invariant subspace $\mathcal{V} \in \mathbb{R}^n$

<u>PR.1</u> If the columns of $V \in \mathbb{R}^{n \times k}$ form a basis for $\mathcal{V}$, there exists a $k \times k$ $\bar{A}$ s.t.

$$AV = V\bar{A}$$

<u>PR.2</u> $\mathcal{V}$ contains at least one eigenvector of A

Proof: Let $V = [v_1 \cdots v_k]$ form a basis for $\mathcal{V}$

then $Av_i \in \mathcal{V}$ implies that $Av_i = \bar{a}_{1i} v_1 + \bar{a}_{2i} v_2 + \cdots \bar{a}_{ki} v_k$

$$= V\bar{a}_i$$

$$\Rightarrow A[v_1 \cdots v_k] = V[\bar{a}_1 \cdots \bar{a}_k]$$

$$\Rightarrow AV = V\bar{A}$$

proof of PR.2

Let $(\lambda, \bar{v})$ be an eigenpair of $\bar{A}$,

then $\bar{A}\bar{v} = \lambda \bar{v}$

$\Rightarrow \quad AV\bar{v} = V\bar{A}\bar{v} = \lambda V \bar{v}$

$\Rightarrow \quad (\lambda, V\bar{v})$ is an eigenpair of A and

$V\bar{v} \in \mathcal{V}$

Thm 12.2 (Eigenvector test for Controllability)

The LTI system (continuous / discrete) (A, B) is controllable iff there is no eigenvector of A^T that is in the kernel of B^T.

proof: $(\Rightarrow)$ (necessary condition)

Assume (A,B) - controllable. To complete the proof by contradiction, assume that there is an eigenvector of A^T that is in the kernel of B^T,

i.e. $\exists\, x \neq 0$ s.t. $A^T x = \lambda x$ and $B^T x = 0$

then $$\mathcal{C}^T x = \cancel{\begin{bmatrix} B^T \\ A^T B^T \\ \vdots \\ (A^{n-1})^T B^T \end{bmatrix} x} = \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^{n-1})^T \end{bmatrix} x$$

$$= \begin{bmatrix} B^T x \\ \lambda B^T x \\ \vdots \\ \lambda^{n-1} B^T x \end{bmatrix} = 0 \quad \Rightarrow \mathcal{C} \text{ is not full rank}$$

$\Rightarrow (A,B)$ - not controllable.

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($\Leftarrow$) (Sufficiency)

We first show that $\ker \mathcal{C}^T$ is A^T-invariant.

Let $x \in \ker \mathcal{C}^T$ ie $\mathcal{C}^T x = 0$ ie $\begin{pmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^{n-1})^T \end{pmatrix} x = 0$

need to show that $Ax \in \ker \mathcal{C}^T$, or

$$\mathcal{C}^T A x = \begin{pmatrix} B^T A^T \\ B^T (A^2)^T \\ \vdots \\ B^T (A^{n-1})^T \\ B^T (A^n)^T \end{pmatrix} x = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ B^T (A^n)^T x \end{pmatrix}$$

but by the Cayley-Hamilton Thm

$$(A^n)^T = \alpha_0 I + \alpha_1 A^T + \alpha_2 (A^2)^T + \cdots \alpha_{n-1} (A^{n-1})^T$$

$$\Rightarrow B^T (A^n)^T x = 0$$

$$\Rightarrow Ax \in \ker C^T$$

$\therefore$ From property 12.2 $\ker \mathcal{C}^T$ contains at least one eigenvector x of A^T ie $\mathcal{C}^T x = 0 \Rightarrow B^T x = 0$

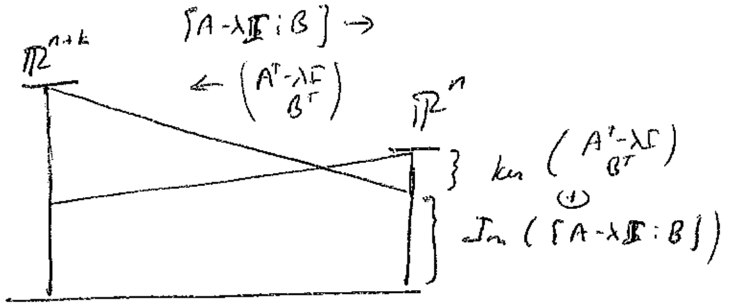
So if there is no eigenvector of A^T in the $\ker B^T$, then there is no $x \neq 0$ s.t. $\mathcal{C}^T x = 0$

$\Rightarrow$ rank $\mathcal{C} = n$ $\Rightarrow$ Controllable.

Thm 12.3 PBH (Popov-Belevitch-Hautus) test for controllability

(A,B) is controllable iff $\operatorname{rank}[A-\lambda I \,\vdots\, B] = n \quad \forall \lambda \in \mathbb{C}$

proof:



$\Rightarrow)$ ~~Since~~ $n = \dim \ker \begin{pmatrix} A^T-\lambda I \\ B^T \end{pmatrix} + \underbrace{\dim \operatorname{Im}(A-\lambda I : B)}_{= \operatorname{rank}(A-\lambda I : B)}$

$\Rightarrow) \quad \dim \ker \begin{pmatrix} A^T-\lambda I \\ B^T \end{pmatrix} = n - \operatorname{rank}(A-\lambda I ; B)$

~~If rank~~

So $\operatorname{rank}(A-\lambda I : B) = n$

$\iff \dim \ker \begin{pmatrix} A^T-\lambda I \\ B^T \end{pmatrix} = 0$

$\iff \ker \begin{pmatrix} A^T-\lambda I \\ B^T \end{pmatrix} = \{ x \in \mathbb{R}^n : A^T x = \lambda x \text{ and } B^T x = 0 \} = \{0\}$

$\iff$ no eigenvectors of A^T are in $\ker B^T$

$\iff (A,B)$ - controllable

12.3 Lyapunov test for Controllability

Consider the LTI system

$$\dot{x} = Ax + Bu \quad , \quad x^{+} = Ax + Bu$$

Thm 12.4 Assume A is a stability matrix

ie $\text{eig}(A)$ in LHP (CT) or unit circle (DT)

Then (A,B)-controllable iff $\exists$ a unique $\overset{PD}{\vee}$ solution to

$$AW + WA^T = -BB^T \quad / \quad AWA^T - W = -BB^T$$

Moreover

$$W = \int_0^{\infty} e^{A\tau} BB^T e^{A^T\tau}\, d\tau = \lim_{t_1 - t_0 \to \infty} W_R(t_0, t_1) \qquad (CT)$$

$$W = \sum_{\tau=0}^{\infty} A^{\tau} BB^T (A^T)^{\tau} = \lim_{t_1 - t_0 \to \infty} W_R(t_0, t_1) \qquad (DT)$$

proof:

($\Leftarrow$) (Sufficiency)

Suppose there exists a unique $\overset{PD}{\vee}$ solution to $AW + WA^T = -BB^T$

We will use the eigenvector test by letting $x \neq 0$ be an eigenvector of A^T, We need to show that $B^T x \neq 0$

$$x^*(AW + WA^T)x = -x^* BB^T x = -\|B^T x\|^2$$

x is ~~complex~~ possibly complex

but $x^*(AW + WA^T)x = (A^T \bar{x})^T W x + x^* W A^T x$

$$= \bar{\lambda}\, x^* W x + \lambda\, x^* W x$$

$$= 2\,\text{Re}\{\lambda\}\, x^* W x$$

< 0 since $W > 0$ and A is a stability matrix.

$\therefore \|B^T x\| \neq 0 \Rightarrow (A,B)$-controllable.

($\Rightarrow$) (Necessity)

Assume (A,B) - controllable

Since A is a stability matrix we have (from lecture 8)

$$W = \int_0^\infty e^{A\tau} BB^T e^{A^T\tau} d\tau \quad \text{is a solution of}$$

$$AW + WA^T = -BB^T$$

but since $BB^T \geq 0$ lecture 8 only guaranteed that

$$W \geq 0$$

but

$$
\begin{aligned}
x^T W x &= x^T \left(\int_0^\infty e^{A\tau} BB^T e^{A^T\tau} d\tau \right) x \\
&= x^T \int_0^\infty \| B^T e^{A^T\tau} \|^2 d\tau \; x \\
&\geq x^T \left(\int_0^1 e^{A\tau} BB^T e^{A^T\tau} d\tau \right) x \\
&= x^T W_R(0,1)\, x \\
&> 0 \quad \text{since } (A,B)\text{-controllable}
\end{aligned}
$$

$$\Rightarrow W > 0$$

We can also show that if $\exists$ a p.d. solution P to

$$AP + PA^T = -BB^T \quad \text{and} \quad (A, B)\text{-controllable},$$

then A is a stability matrix.

<u>proof</u>: Let (λ, x) be an eigen pair of A^T.

Then (A, B)-controllable $\Rightarrow$ $B^T x \neq 0$

so

$$\begin{array}{ccc} x^*(AP + PA^T)x & = & x^* B B^T x \\ \| & & \| \\ 2\,\mathrm{Re}\{\lambda\}\underbrace{x^* P x}_{\substack{>0 \\ \text{by} \\ \text{assumption}}} & = & \underbrace{-\|B^T x\|^2}_{\text{non-zero by controllability}} \end{array}$$

$\therefore \quad \mathrm{Re}\{\lambda\} < 0$

12.4 Feedback Stabilization

<u>Fact</u>: (A, B) - controllable $\Leftrightarrow$ $(-\mu I - A, B)$ - controllable

<u>proof</u>: since $A^T x = \lambda x \Leftrightarrow (-\mu I - A)^T x = -\mu x - \lambda x$

$$= -(\mu + \lambda) x$$

implies that A^T and $(-\mu I - A)^T$ have the same eigenvectors, and $B^T x \neq 0$

$\therefore$ we can choose μ to make $(-\mu I - A)$ a stability matrix

$\therefore$ (A, B) - controllable implies $\exists$ pd W s.t.

$$(-\mu I - A) W + W(-\mu I - A)^T = -BB^T$$

$$\Leftrightarrow \quad AW + WA^T - BB^T = -2\mu W$$

let $P = W^{-1}$ and pre- and post- multiply by P:

$$\Leftrightarrow \quad PA + A^T P - PBB^T P = -2\mu P$$

$$\Leftrightarrow \quad P(A - \tfrac{1}{2}BB^T P) + (A - \tfrac{1}{2}BB^T P)^T P = -2\mu P$$

or $\quad P(A - BK) + (A - BK)^T P = -2\mu P$

where $K = \frac{1}{2} B^T P$

$\therefore$ $A - BK$ is a stability matrix and $u = -Kx$ asymptotically stabilizes the system

Thm 12.6 When (A,B)-controllable,
for every $\mu > 0$, it is possible to find
a controller ~~used~~ $u = -Kx$ that places all of the
eigenvalues of $\dot{x} = (A - BK)x$ on $\{ Re\{s\} \le -\mu \}$