Lecture 13 Controllable Decompositions

13.1 Similarity Transformation

Given $\dot{x} = Ax + Bu$

and the change of variables $\bar{x} = T^{-1}x$ (for any invertible T)

then
$$\dot{\bar{x}} = T^{-1}AT\bar{x} + T^{-1}Bu$$

Define $\bar{A} = T^{-1}AT$, $\bar{B} = T^{-1}B$

Thm 13.1
$$(A,B)\text{ -controllable} \iff (\bar{A},\bar{B})\text{ -controllable}$$

proof: Let $\mathcal{C} = [B \; AB \; \cdots \; A^{n-1}B]$

then
$$\bar{\mathcal{C}} = [\bar{B} \; \bar{A}\bar{B} \; \cdots \; \bar{A}^{n-1}\bar{B}] = [T^{-1}B \;\; T^{-1}ATT^{-1}B \;\; T^{-1}ATT^{-1}ATT^{-1}B \;\; \cdots \;\;]$$
$$= T^{-1}[B \;\; AB \;\; A^2B \cdots A^{n-1}B]$$
$$= T^{-1}\mathcal{C}$$

Given the rank formula

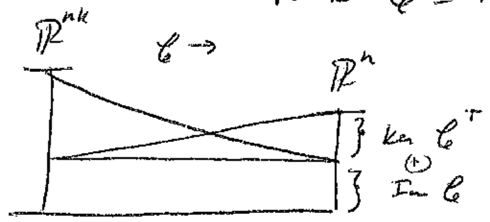
$$\operatorname{rank}(M) + \operatorname{rank}(N) - n \le \operatorname{rank}(\underbrace{MN}_{k\times n \, n\times p}) \le \min\{\operatorname{rank}(M), \operatorname{rank}(N)\}$$

Since $\operatorname{rank}(T^{-1}) = n$, we have that $\operatorname{rank}\mathcal{C} = \operatorname{rank}\bar{\mathcal{C}}$

13.2 Controllable Decomposition

What if the system is not controllable, i.e.,

$$\text{rank } \mathcal{C} = \bar{n} < n$$



SVD:

$$\underbrace{\mathcal{C}}_{n \times nk} = [\underset{n \times \bar{n}}{U_1}, \underset{n \times (n-\bar{n})}{U_2}] \begin{bmatrix} \overset{\bar{n} \times \bar{n}}{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} V_1^H \\ V_2^H \end{bmatrix}}_{nk \times nk}$$

where $\operatorname{span}(U_1) = \operatorname{Im} \mathcal{C}$

$\operatorname{span}(U_2) = \ker \mathcal{C}^T$

and $\operatorname{Im} U = \mathbb{R}^n$ where $U = [U_1, U_2]$

~~Since~~ From previous lecture we saw that $\operatorname{Im} \mathcal{C}$ is A-invariant

$$\Rightarrow \quad AU_1 = U_1 A_c = [U_1 \ U_2] \begin{bmatrix} A_c \\ 0 \end{bmatrix}$$

$\nwarrow$ linear combination of columns of U_1

Also since $\mathcal{C} = [B \ AB \ \cdots \ A^{n-1}B]$ we have $\operatorname{span}(B) \subseteq \operatorname{Im} \mathcal{C}$

$$\Rightarrow \quad B = U_1 B_c \quad \Rightarrow \quad B = [U_1 \ U_2] \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$

$$A[U_1 \ U_2] = [U_1 \ U_2] \begin{bmatrix} A_c & A_{12} \\ 0 & \underbrace{A_u}_{\text{other stuff}} \end{bmatrix}$$

$$\Rightarrow \quad [U_1 \ U_2]^{-1} A [U_1 \ U_2] = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}$$

Let $U = [U_1 \ U_2]$ then

$$U^{-1} A U = \begin{pmatrix} A_c & A_{12} \\ 0 & A_u \end{pmatrix} \qquad U^{-1} B = \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$

Thm 13.2 For every LTI system (A, B), there is a similarity transformation T s.t.

$$T^{-1}AT = \begin{pmatrix} A_c & A_{12} \\ 0 & A_u \end{pmatrix}, \quad T^{-1}B = \begin{pmatrix} B_c \\ 0 \end{pmatrix}$$

s.t.

1. The controllable subspace of the transformed system is

$$\operatorname{Im} \bar{\mathcal{C}} = \operatorname{Im} \begin{pmatrix} I_{\bar{n}\times\bar{n}} \\ 0 \end{pmatrix}$$

2. (A_c, B_c) - controllable.

proof:

1) $$\bar{\mathcal{C}} = \left[\begin{pmatrix} B_c \\ 0 \end{pmatrix} \quad \begin{pmatrix} A_c & A_{12} \\ 0 & A_u \end{pmatrix}\begin{pmatrix} B_c \\ 0 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} A_c & A_{12} \\ 0 & A_u \end{pmatrix}^{n-1}\begin{pmatrix} B_c \\ 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} B_c & A_c B_c & \cdots & A_c^{n-1} B_c \\ 0 & 0 & & 0 \end{pmatrix} \} \leftarrow 1^{st}\ \bar{n} \text{ rows}$$

Since $\bar{\mathcal{C}}$ is rank $\bar{n}$ the first $\bar{n}$ rows are linearly indep.

$$\Rightarrow \operatorname{Im}\{\bar{\mathcal{C}}\} = \operatorname{Im}\begin{pmatrix} I_{\bar{n}\times\bar{n}} \\ 0 \end{pmatrix}$$

2) By Cayley-Hamilton, the ~~$\mathcal{B}$~~

$$\operatorname{rank}\begin{pmatrix} B_c & A_c B_c & \cdots & A_c^{n-1} B_c \end{pmatrix} = \operatorname{rank}\begin{pmatrix} B_c & A_c B_c & \cdots & A_c^{\bar{n}-1} B_c \end{pmatrix} = \bar{n}$$

13.3 Block Diagram

In the transformed variables $\bar{x} = T^{-1}x = \begin{pmatrix} x_c \\ x_u \end{pmatrix}$

we have

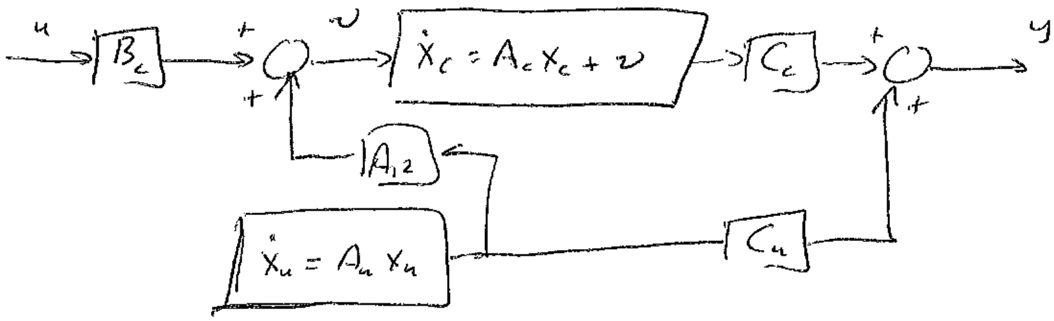
$$\begin{pmatrix} \dot{x}_c \\ \dot{x}_u \end{pmatrix} = \begin{pmatrix} A_c & A_{12} \\ 0 & A_u \end{pmatrix} \begin{pmatrix} x_c \\ x_u \end{pmatrix} + \begin{pmatrix} B_c \\ 0 \end{pmatrix} u$$

or

$$\dot{x}_c = A_c x_c + A_{12} x_u + B_c u$$

$$\dot{x}_u = A_u x_u$$

$$y = \begin{pmatrix} C_c & C_u \end{pmatrix} \begin{pmatrix} x_c \\ x_u \end{pmatrix} + Du$$



u does not influence x_u

13.4 Transfer Function

Transfer function is the same for all state space realizations.

the

$$T(s) = C(sI - A)^{-1}B + D$$

$$= \begin{pmatrix} C_c & C_u \end{pmatrix} \left(\begin{pmatrix} sI & 0 \\ 0 & sI \end{pmatrix} - \begin{pmatrix} A_c & A_{12} \\ 0 & A_u \end{pmatrix} \right)^{-1} \begin{pmatrix} B_c \\ 0 \end{pmatrix} + D$$

$$= \begin{pmatrix} C_c & C_u \end{pmatrix} \begin{pmatrix} (sI - A_c) & -A_{12} \\ 0 & (sI - A_u) \end{pmatrix}^{-1} \begin{pmatrix} B_c \\ 0 \end{pmatrix} + D$$

$$= \begin{pmatrix} C_c & C_u \end{pmatrix} \begin{pmatrix} (sI - A_c)^{-1} & * \\ 0 & (sI - A_u)^{-1} \end{pmatrix} \begin{pmatrix} B_c \\ 0 \end{pmatrix} + D$$

$$= \begin{pmatrix} C_c & C_u \end{pmatrix} \begin{pmatrix} (sI - A_c)^{-1} B_c \\ 0 \end{pmatrix} + D$$

$$= C_c (sI - A_c)^{-1} B_c + D$$