Lecture 14 stabilizability

Any LTI system ~~can be transformed~~ is algebraically equivalent, through a similarity transformation, to the following form

$$\begin{pmatrix} \dot{x}_c / x_c^+ \\ \dot{x}_u / x_u^+ \end{pmatrix} = \begin{pmatrix} A_c & A_{12} \\ 0 & A_u \end{pmatrix} \begin{pmatrix} x_c \\ x_u \end{pmatrix} + \begin{pmatrix} B_c \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} C_c & C_u \end{pmatrix} \begin{pmatrix} x_c \\ x_u \end{pmatrix} + Du$$

$x_c \in \mathbb{R}^{\bar{n}}$, $x_u \in \mathbb{R}^{n-\bar{n}}$, $u \in \mathbb{R}^k$, $y \in \mathbb{R}^m$

Def 14.1 The system (A,B) is stabilizable, if $n = \bar{n}$ or A_u is a stability matrix.

14.2 Eigenvector Test for Stabilizability

Thm 14.1

~~The ...~~

1. $$\dot{x} = Ax + Bu$$

is stabilizable iff every eigenvector of A^T corresponding to an eigenvalue with positive or zero real part, is not in the kernel of B^T

2. $$x^+ = Ax + Bu$$

is stabilizable iff every eigenvector of A^T corresponding to an ~~eigen~~ eigenvalue with magnitude ≥ 1, is not in $\ker B^T$.

proof: Use similarity transformation to transform to controllable decomposition

$$\bar{A} = \begin{pmatrix} A_c & A_{12} \\ 0 & A_u \end{pmatrix} = T^{-1}AT \qquad \bar{B} = \begin{pmatrix} B_c \\ 0 \end{pmatrix} = T^{-1}B$$

$\Rightarrow$

Assume the system is stabilizable but that there is an unstable eigenvector in null space of B^T

ie (λ, x) - eigen pair s.t. $A^T x = \lambda x$ and $B^T x = 0$

$$\Leftrightarrow \quad (T^{-1}\bar{A}T)^T x = \lambda x \quad \text{and} \quad (T^{-1}B)^T x = 0$$

$$\Leftrightarrow \quad \bar{A}(T^T x) = \lambda (T^T x) \text{ and } \bar{B}(T^T x) = 0$$

$$\Leftrightarrow \quad \begin{pmatrix} A_c^T & 0 \\ A_{12}^T & A_u^T \end{pmatrix} \begin{pmatrix} x_c \\ x_u \end{pmatrix} = \lambda \begin{pmatrix} x_c \\ x_u \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} B_c^T & 0 \end{bmatrix} \begin{pmatrix} x_c \\ x_u \end{pmatrix} = 0$$

where ~~the~~ $\begin{pmatrix} x_c \\ x_u \end{pmatrix} = T^T x \neq 0$

Since (A_c, B_c) – controllable and

$$A_c^T x_c = \lambda x_c \quad \text{and} \quad B_c^T x_c = 0$$

then $x_c = 0$ (ie not an eigenvector)

$\Rightarrow$ $A_u^T x_u = \lambda x_u \Rightarrow \lambda$ is an (unstable) eigenvalue of A_u, which contradicts stabilizability.

$\Leftarrow$ Assume every unstable eigenvector of A^T is not in $\ker B^T$ and assume system is not stabilizable.

Not stabilizable $\Rightarrow \exists \lambda \in \overline{RHP}$ s.t. $A_u^T x_u = \lambda x_u$, $x_u \neq 0$

But
$$\begin{pmatrix} A_c^T & 0 \\ A_{12}^T & A_u^T \end{pmatrix} \begin{pmatrix} 0 \\ x_u \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ x_u \end{pmatrix}$$

and
$$\bar{B}^T \begin{pmatrix} 0 \\ x_u \end{pmatrix} = \begin{pmatrix} B_c^T & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_u \end{pmatrix} = 0 \quad \text{for} \begin{pmatrix} 0 \\ x_u \end{pmatrix} \neq 0$$

$\Rightarrow \begin{pmatrix} 0 \\ x_u \end{pmatrix}$ is an unstable eigenvector of $\bar{A}^T$ and $\begin{pmatrix} 0 \\ x_u \end{pmatrix} \in \ker \bar{B}^T$

contradiction since similarity transform preserves controllability properties.

PBH Test for Stabilizability

Thm 14.2

1. the cont-time LTI system is stabilizable

$\iff$ rank $[A-\lambda I \,\vdots\, B] = n$ $\forall \lambda$ in closed RHP

2. the disc-time LTI system is stabilizable

$\iff$ rank $[A-\lambda I \,\vdots\, B] = n$ $\forall \lambda$ outside open disk.

14.4 Lyapunov Test for Stabilizability

Thm 14.3 The LTI system

$$\dot{x} = Ax + Bu \quad / \quad x^+ = Ax + Bu$$

is stabilizable iff there exists a p.d. P s.t.

$$AP + PA^T - BB^T < 0 \quad / \quad APA^T - P - BB^T < 0$$

proof (Discrete-time)

($\Leftarrow$) Suppose that $APA^T - P - BB^T < 0$ and let (λ, x) be an unstable eigenpair of A^T

Then $x^*(APA^T - P)x < x^* BB^T x$

$\iff$ $(A^T x)^* P (A^T x) - x^* P x < \|B^T x\|^2$

$\iff$ $|\lambda|^2 x^* P x - x^* P x < \|B^T x\|^2$

$\iff$ $(|\lambda|^2 - 1)\, x^* P x < \|B^T x\|^2$

Since P is pd and $x \neq 0$ we have

$$0 \le (|\lambda|^2 - 1)\, x^* P x < \|B^T x\|^2 \Rightarrow x \notin \ker B^T$$

$\Rightarrow$ stabilizable.

⇒ Assume (A,B) - stabilizable

Let T be similarity transform leading to

$$\bar{A} = T^{-1} A T = \begin{pmatrix} A_c & A_{12} \\ 0 & A_u \end{pmatrix} \qquad \bar{B} = \begin{bmatrix} B_c \\ 0 \end{bmatrix} = T^{-1} B$$

(A_c, B_c) - controllable ⇒ $\exists\, P_c$ s.t.

$$A_c P_c A_c^T - P_c - B_c B_c^T = -Q_c < 0$$

A_u - stable ⇒ $\exists\, P_u$ s.t.

$$A_u P_u A_u^T - P_u = -Q_u < 0$$

Let $\bar{P} = \begin{pmatrix} A_c & 0 \\ 0 & \rho A_u \end{pmatrix}$

Then

$$\bar{A} \bar{P} \bar{A}^T - \bar{P} - \bar{B} \bar{B}^T$$

$$= \begin{pmatrix} A_c & A_{12} \\ 0 & A_u \end{pmatrix} \begin{pmatrix} P_c & 0 \\ 0 & \rho P_u \end{pmatrix} \begin{pmatrix} A_c^T & 0 \\ A_{12}^T & A_u^T \end{pmatrix} - \begin{pmatrix} P_c & 0 \\ 0 & \rho P_u \end{pmatrix} - \begin{pmatrix} B_c \\ 0 \end{pmatrix} \begin{pmatrix} B_c^T & 0 \end{pmatrix}$$

$$= \begin{pmatrix} A_c P_c A_c^T + \rho A_{12} P_u A_{12}^T - P_c - B_c B_c^T & \rho A_{12} P_u A_c^T \\ \rho A_u P_u A_{12}^T & \rho A_u P_u A_u^T - \rho P_u \end{pmatrix}$$

$$= \begin{pmatrix} -Q_c + \rho A_{12} A_u A_{12}^T & \rho A_{12} P_u A_c^T \\ \rho A_u P_u A_u^T & -\rho Q_u \end{pmatrix}$$

will be negative definite for ρ small enough

14.5 Feedback stabilization based on Lyapunov Test

For cont-time system $\dot{x} = Ax + Bu$

(A,B) - stabilizable $\Rightarrow$ $\exists$ pd P s.t.

$$AP + PA^T - BB^T < 0$$

$$\Leftrightarrow \quad AP + PA^T - \tfrac{1}{2} BB^T P^{-1} P - \tfrac{1}{2} P P^{-1} BB^T < 0$$

$$\Leftrightarrow \quad (A - \tfrac{1}{2} BB^T P^{-1}) P + P(A - \tfrac{1}{2} BB^T P^{-1})^T < 0$$

$$\Leftrightarrow \quad (A - Bk)P + P(A-Bk)^T < 0$$

where $K = -\tfrac{1}{2} B^T P^{-1}$

$$\Leftrightarrow \quad PP^{-1}(A-Bk)P + P(A-BK)^T P^{-1} P < 0$$

$$\Leftrightarrow \quad P\left[P^{-1}(A-Bk) + (A-Bk)^T P^{-1}\right]P < 0$$

$$\Leftrightarrow \quad Q(A-Bk) + (A-Bk)^T Q < 0$$

$\therefore$ $A-BK$ is a stability matrix by the Lyapunov thm.

Thm 14.4 (A,B) - stabilizable iff $\exists K$ s.t. $u = -kx$ makes the closed-loop system $\dot{x} = (A-Bk)x$ asymptotically stable.

14.6 Eigenvalue Assignment: pole placement.

Thm 14.6 If $\dot{x}/x^+ = Ax + Bu$ is controllable.

Given any set of n complex #s (appearing in complex conjugate pairs), $\exists K$ s.t. $A-Bk$ has eigenvalues equal to λ_i.