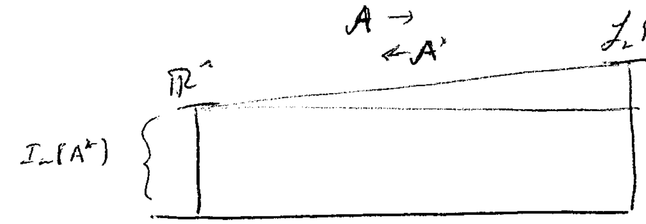
Lecture 15 Observability

The solution to the system $\dot{x} = A(t)x$, $x(t_0) = x_0$
$y = C(t)x$

is

$$y(t) = \underbrace{C(t)\,\Phi(t,t_0)}\,x_0$$

This is a linear operator that maps $\mathbb{R}^n$ to $\mathcal{L}_2[t_0,t_f]$



The adjoint $\mathcal{A}^*$ mapping $\mathcal{L}_2[t_0,t_f]$ to $\mathbb{R}^n$ is defined by

$$\left\langle z(t),\ \mathcal{A}[x_0](t)\right\rangle_{\mathcal{L}_2[t_0,t_f]} = \left\langle \mathcal{A}^*[z(t)],\ x_0\right\rangle_{\mathbb{R}^n}$$

$$\Leftrightarrow \int_{t_0}^{t_f} x_0^T\,\Phi^T(\tau,t_0)\,C^T(\tau)\,z(\tau)\,d\tau = x_0^T\mathcal{A}^*[z(t)]$$

$$\Leftrightarrow x_0^T\left(\int_{t_0}^{t_f}\Phi^T(\tau,t_0)\,C^T(\tau)\,z(\tau)\,d\tau\right) = x_0^T\mathcal{A}^*[z(t)]$$

$$\Rightarrow \mathcal{A}^*[z(t)] = \int_{t_0}^{t_f}\Phi^T(\tau,t_0)\,C^T(\tau)\,z(\tau)\,d\tau$$

Note that $I_m[\mathcal{A}^*] = I[\mathcal{A}^*\mathcal{A}]$

where $\mathcal{A}^*\mathcal{A} = \int_{t_0}^{t_f}\Phi^T(\tau,t_0)\,C^T(\tau)\,C(\tau)\,\Phi(\tau,t_0)\,d\tau \triangleq W_0(t_0,t_f)$

The observability Gramian.

Note that for the more general case where

$$y(t) = C(t)\Phi(t,t_0)x_0 + \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

we have $\tilde{y}(t) = C(t)\Phi(t,t_0)x_0$ where

$$\tilde{y}(t) = y(t) - \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

leading to the same linear operator A.

~~From EE 671 we can make the~~

Def: The $\ker(A) = \ker(C(t)\Phi(t,t_0))$ ie, all states $x_0 \in \mathbb{R}^n$ s.t. $C(t)\Phi(t,t_0)x_0 = 0 \quad \forall t \in [t_0, t_f]$ is called the Unobservable subspace $\mathcal{UO}[t_0,t_f]$.

Def: the system is said to be observable if $\ker(A) = \{0\}$, ie $\mathcal{UO}[t_0,t_f] = \{0\}$

From EE 671 we can conclude that

Corollary 15.1 : The system is observable iff $\operatorname{rank}(W_0[t_0,t_f]) = n$

Thm 15.1 : $\mathcal{UO}[t_0,t_f] = \ker(W_0(t_0,t_f))$

proof: Since $W_0 = W_0^T$ we have that

$$\ker(W_0) = \ker(W_0^T)$$

we also know that $\ker(W_0^T) = \ker(A^*A)$

$$= \ker(A)$$

$$= \mathcal{UO}[t_0,t_f]$$

Application

Given $\dot{x} = A(t)x + B(t)u$, $y(t) = C(t)x + D(t)u$, $x(t_0) = x_0$

Suppose that the problem is to find the initial condition x_0 by observing $y(t)$ and $u(t)$ over the interval $[t_0, t_f]$

We have that

$$y(t) = C(t)\Phi(t, t_0)x_0 + \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau) + D(t)u(t)$$

Let $\tilde{y}(t) = y(t) - \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau) - D(t)u(t)$

then $\tilde{y}(t) = C(t)\Phi(t,t_0)x_0$ where $\tilde{y}(t)$ is assumed known

$= \mathcal{A}[x_0]$

Apply $\mathcal{A}^*$ to $\tilde{y}(t)$:

$$\mathcal{A}^*[\tilde{y}(t)] = \int_{t_0}^{t_f} \Phi^T(\tau, t_0)C^T(\tau)\tilde{y}(\tau)\,d\tau = \mathcal{A}^*\mathcal{A}x_0 = W_0[t_0, t_f]x_0$$

If the system is observable, then $\operatorname{rank}(W_0(t_0,t_f)) = n$

$$\therefore \quad x_0 = W_0^{-1}[t_0, t_f]\int_{t_0}^{t_f} \Phi^T(\tau, t_0)C^T(\tau)\tilde{y}(\tau)\,d\tau$$

What if we are interested in reconstructing the current state, as opposed to the initial state?

- Lets do discrete-time case when $A(t)$ is invertible.

We can write

$$y(t) = C(t)\,\Phi(t, t_f)\,x_f + \sum_{\tau = t_f}^{t} C(t)\,\Phi(t,\tau) B(\tau) u(\tau)\, d\tau + D(t) u(t)$$

backward in time

$$t_0 \le t \le t_f$$

Again let $\tilde{y}(t) = y(t) - \sum_{\tau = t_f}^{t} C(t)\,\Phi(t,\tau) B(\tau) u(\tau)\, d\tau + D(t) u(t)$

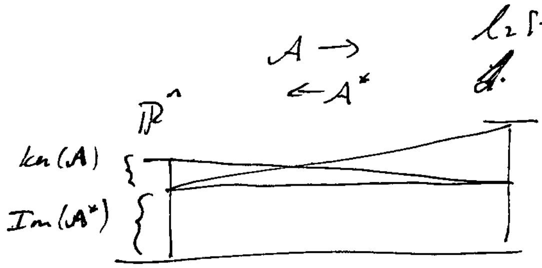
and assume $\tilde{y}(t)$ is known, for $t_0 \le t \le t_f$

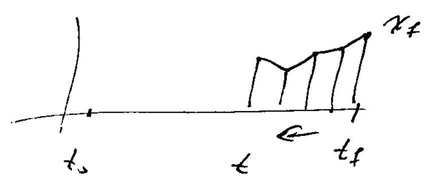
Find x_f

Since $\tilde{y}(t) = C(t)\,\Phi(t, t_f)\, x_f$

Define the operator $\mathcal{A}[x_f] \triangleq C(t)\,\Phi(t, t_f)\, x_f$

starting with x_f, propagates backward in time.





$\mathcal{A}^*$ is defined by

$$\langle z(t), \mathcal{A}[x_f] \rangle_{l_2[t_0, t_f]} = \langle \mathcal{A}^*[z(t)], x_f \rangle_{\mathbb{R}^n}$$

where

$$\langle z(t), \mathcal{A}\{x_f\} \rangle_{\ell_2[t_0,t_f]} = \langle z(t), C(t)\Phi(t,t_f)x_f \rangle_{\ell_2}$$

$$= \sum_{\tau=t_0}^{t_f} x_f^T \Phi^T(t,t_f) C^T(t) z(t)$$

and

$$\langle \mathcal{A}^*\{z(t)\}, x_f \rangle_{\mathbb{R}^n} = x_f^T \mathcal{A}^*\{z(t)\}$$

$$\Rightarrow \mathcal{A}^*\{z(t)\} = \sum_{\tau=t_0}^{t_f} \Phi^T(t,t_f) C^T(t) z(t)$$

Since $\quad \mathrm{Im}(\mathcal{A}^*) = \mathrm{Im}(\mathcal{A}^*\mathcal{A})$

where $\quad \mathcal{A}^*\mathcal{A} = \sum_{\tau=t_0}^{t_f} \Phi^T(\tau,t_f) C^T(\tau) C(\tau) \Phi(\tau,t_f) \triangleq W_{cn}(t_0,t_f)$

<u>Def</u>: The system is constructable if ~~$\mathrm{rank}(\mathcal{A}^*\mathcal{A}) = \mathrm{rank}(W_{cn}(t_0,t_f)) = n$~~

$$\ker(\mathcal{A}) = \{0\}$$

<u>Thm</u>: The system is constructable iff $\mathrm{rank}(W_{cn}(t_0,t_f)) = n$

Reconstructing the current state from past outputs:

given $\tilde{y}(t)$ known, for $t \in [t_0, t_f]$

where $\tilde{y}(t) = C(t)\,\Phi(t, t_f)\, x_f$

Apply A^* to both sides:

$$\int_{\tau = t_0}^{t_f} \Phi^T(\tau, t_f)\, C^T(\tau)\, \tilde{y}(\tau)\, d\tau = \left(W_{cn}(t_0, t_f)\right) x_f$$

If the system is constructable then

$$x_f = \left(W_{cn}(t_0, t_f)\right)^{-1} \int_{\tau = t_0}^{t_f} \Phi^T(\tau, t_f)\, C^T(\tau)\, y(\tau)\, d\tau$$

— Batch processing to reconstruct state.

15.8 Duality

For LTI systems:

$$\dot{x} = Ax + Bu \quad , \quad y = Cx + Du$$

$$W_O(t_0, t_f) = \int_{t_0}^{t_f} e^{A^T(\tau - t_0)} C^T C \, e^{A(\tau - t_0)} \, d\tau$$

Recall also that the controllability grammian is

$$W_C(t_0, t_f) = \int_{t_0}^{t_f} e^{A(\tau - t_0)} BB^T e^{A^T(\tau - t_0)} \, d\tau$$

where we have

$$(A, B) \text{ is controllable} \iff \text{rank}(W_c) = n$$

and (A, C) - observable $\iff \text{rank}(W_o) = n$

Note that if A and B in W_c is replaced with A^T, C^T, then the result is W_O, ie

$$W_c(A, B) = W_o(A^T, B^T)$$

and $W_o(A, C) = W_c(A^T, C^T)$

Thm 15.5 (Duality for LTI systems)

(A, B) - controllable $\iff$ (A^T, B^T) - ~~observable~~ observable

(A, C) - observable $\iff$ (A^T, C^T) - controllable

Thm 15.6

(A, B) - reachable $\iff$ (A^T, B^T) - constructable

(A, C) - constructable $\iff$ (A^T, C^T) - reachable.

15.9

We can use the duality result to get tests for observability.

Define $\mathcal{C}(A,B) = [B \;\; AB \;\; A^2B \;\cdots\; A^{n-1}B]$

Then $\mathcal{C}(A^T, C^T] = [C^T \;\; A^TC^T \;\; (A^T)^2C^T \;\cdots\; (A^T)^{n-1}C^T)$

and $\mathcal{C}^T [A^T, C^T) = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$

Define $\mathcal{O}(A,C) \triangleq \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$ as the observability matrix.

Thm 15.7

(A,C) - observable iff $\operatorname{rank} \mathcal{O}(A,C) = n$

Alternative proof

Consider $\dot{x} = Ax$, $y = Cx$ $\quad x(0) = x_0$

then $y(t) = Ce^{At}x_0$

Also note that
$$\dot{y}(t) = CAe^{At}x_0$$
$$\ddot{y}(t) = CA^2e^{At}x_0$$
$$\vdots$$
$$y^{(n-1)}(t) = CA^{n-1}e^{At}x_0$$

$\therefore$ evaluating at time $t=0$ gives

$$\begin{pmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{pmatrix} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} x_0 = \mathcal{O}(A,C)\, x_0$$

So if (A,C) - observable then $x_0 = \mathcal{O}^{-1}(A,C) \begin{pmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{pmatrix}$

Thm 15.8 The system is observable iff no eigenvector of A is in the $\ker(C)$

Thm 15.9 PBH - test

(A, C) - observable iff $\operatorname{rank}\begin{pmatrix} A-\lambda I \\ C \end{pmatrix} = n \quad \forall \lambda \in \mathbb{C}$

Thm 15.10 Assume A - stability matrix

(A, C) - observable iff there exists a unique p.d. W s.t.

$$A^T W + WA = -C^T C \quad / \quad A^T W A - W = -C^T C$$

$$W = \int_0^\infty e^{A^T\tau} C^T C e^{A\tau} d\tau \qquad W = \sum_{\tau=0}^{\infty} (A^T)^\tau C^T C A^\tau$$