Lecture 16 Output Feedback

Observable decomposition

Thm 16.2 Observable decomposition

For every LTI system $\dot{x} = Ax + Bu$, $y = Cx + Du$

there is a similarity transformation T s.t.

$$T^{-1}AT = \begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix}, \quad CT = [C_o \;\; 0]$$

where (A_o, C_o) - observable

proof: By duality since for (A^T, C^T, B^T) there exist a T

s.t. $$T^{-1}A^T T = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}, \quad T^{-1}C^T = \begin{pmatrix} C_o \\ 0 \end{pmatrix}, \quad B^T T = (B_o \;\; B_{\bar{o}})$$

Note that the SVD of $\underbrace{\mathcal{O}(A,C)}_{mn \times n} = [U_1 \;\; U_2] \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$

V_1 spans the observable subspace and

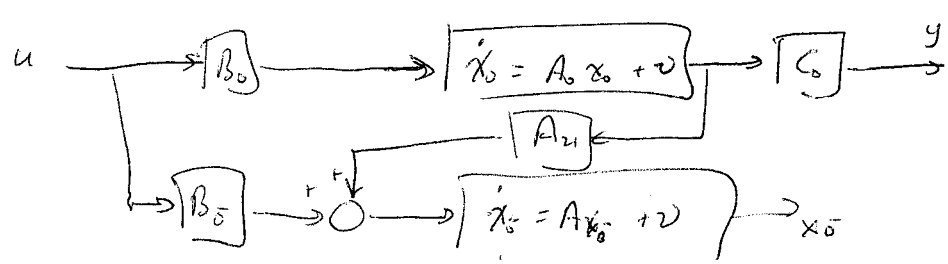
V_2 spans the unobservable subspace.

The transformed system can be written as

$$\begin{bmatrix} \dot{x}_o \\ \dot{x}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix} u, \quad y = [C_o \;\; 0] \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + Du$$

or $\dot{x}_o = A_o x_o + B_o u$, $y = C_o x_o$ $\leftarrow$ no $x_{\bar{o}}$

$\dot{x}_{\bar{o}} = A_{21} x_o + A_{\bar{o}} x_{\bar{o}} + B_{\bar{o}} u$



$x_{\bar{o}}$ is unobservable.

16.2 Kalman Decomposition

We saw that $\exists$ a similarity transform $T_c = [V_c \;\; V_{\bar c}]$ $\begin{bmatrix} x_c \\ x_{\bar c}\end{bmatrix} = T_c^{-1} x$

s.t.

$$\begin{bmatrix} \dot x_c \\ \dot x_{\bar c}\end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar c}\end{bmatrix}\begin{bmatrix} x_c \\ x_{\bar c}\end{bmatrix} + \begin{bmatrix} B_c \\ 0\end{bmatrix} u \qquad y = [C_c \;\; C_{\bar c}]\begin{bmatrix} x_c \\ x_{\bar c}\end{bmatrix} + Du$$

Also $\exists$ a similarity transform $T_o = [V_o \;\; V_{\bar o}]$ ~~s.t.~~ $\begin{bmatrix} x_o \\ x_{\bar o}\end{bmatrix} = T_o^{-1} x$

$$\begin{bmatrix} \dot x_o \\ \dot x_{\bar o}\end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar o}\end{bmatrix}\begin{bmatrix} x_o \\ x_{\bar o}\end{bmatrix} + \begin{bmatrix} B_o \\ B_{\bar o}\end{bmatrix} u \;, \quad y = [C_o \;\; 0]\begin{bmatrix} x_o \\ x_{\bar o}\end{bmatrix} + Du$$

Now, using T_c and T_o, find

$$T = [V_{co} \;\; V_{c\bar o} \;\; V_{\bar c o} \;\; V_{\bar c\bar o}]$$

where

~~$\text{span } V_{co} = \text{span}(\mathcal{C}) \cap \text{span}(\mathcal{O}^T)$~~

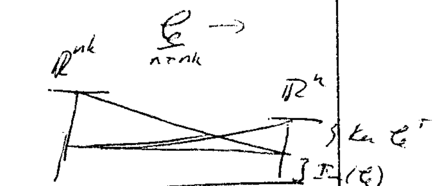
~~$\text{span } V_{\bar c o} = \text{span}(\mathcal{C}) A$~~

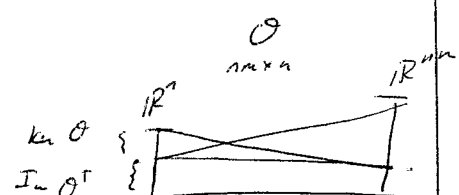
$\text{span } V_{co} = \text{Im}(\mathcal{C}) \cap \text{Im}(\mathcal{O}^T)$

$\text{span } V_{c\bar o} = \text{Im}(\mathcal{C}) \cap \ker(\mathcal{O})$

$\text{span } V_{\bar c o} = \ker(\mathcal{C}^T) \cap \text{Im}(\mathcal{O}^T)$

$\text{span } V_{\bar c\bar o} = \ker(\mathcal{C}^T) \cap \ker(\mathcal{O})$





and let

$$\begin{bmatrix} x_{co} \\ x_{c\bar o} \\ x_{\bar c o} \\ x_{\bar c\bar o}\end{bmatrix} = T^{-1} x$$

Then

$$\begin{bmatrix} \dot x_{co} \\ \dot x_{c\bar o} \\ \dot x_{\bar c o} \\ \dot x_{\bar c\bar o}\end{bmatrix} = \begin{bmatrix} A_{co} & 0 & A_{x o} & 0 \\ A_{c x} & A_{c\bar o} & A_{xx} & A_{x\bar o} \\ 0 & 0 & A_{\bar c o} & 0 \\ 0 & 0 & A_{\bar c x} & A_{\bar c\bar o}\end{bmatrix}\begin{bmatrix} x_{co} \\ x_{c\bar o} \\ x_{\bar c o} \\ x_{\bar c\bar o}\end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar o} \\ 0 \\ 0\end{bmatrix} u$$

$$y = [C_{co} \;\; 0 \;\; C_{\bar c o} \;\; 0]\begin{bmatrix} x_{co} \\ x_{c\bar o} \\ x_{\bar c o}\end{bmatrix} + Du$$

where

1. $\left(\begin{bmatrix} A_{co} & 0 \\ A_{cx} & A_{c\bar{o}} \end{bmatrix}, \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \end{bmatrix} \right)$ — controllable

2. $\left(\begin{bmatrix} A_{co} & A_{xo} \\ 0 & A_{\bar{c}o} \end{bmatrix}, \begin{bmatrix} C_{co} & C_{\bar{c}o} \end{bmatrix} \right)$ — observable

3. (A_{co}, B_{co}, C_{co}) is both observable and controllable

4. $C(sI-A)^{-1}B+D \quad = \quad C_{co}(sI-A_{co})^{-1}B_{co}+D$

$\uparrow$ transfer matrix of original system

$\uparrow$ transfer matrix of the observable and controllable system.

<u>proof of 4:</u>

$$C(sI-A)^{-1}B = \begin{pmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{pmatrix} \begin{pmatrix} sI-A_{co} & 0 & -A_{xo} & 0 \\ -A_{cx} & sI-A_{c\bar{o}} & -A_{xx} & -A_{x\bar{o}} \\ 0 & 0 & sI-A_{\bar{c}o} & 0 \\ 0 & 0 & -A_{\bar{c}x} & sI-A_{\bar{c}\bar{o}} \end{pmatrix}^{-1} \begin{pmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{pmatrix} \begin{pmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{pmatrix} \begin{pmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} C_{co}G_{11} + C_{\bar{c}o}G_{31} \,;\, C_{co}G_{12} + C_{\bar{c}o}G_{32} \,;\, * \,,\, * \end{pmatrix} \begin{pmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{pmatrix}$$

$$= C_{co}G_{11}B_{co} + C_{\bar{c}o}G_{31}B_{co} + C_{co}G_{12}B_{c\bar{o}} + C_{\bar{c}o}G_{32}B_{c\bar{o}}$$

from the formula $A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)}$ ← cofactor transpose, we see that

$G_{31} = 0$

$G_{12} = 0 \qquad$ and $\qquad G_{11} = (sI-A_{co})^{-1}$

16.3 Detectability

Any LTI system is algebraically equivalent to

$$\begin{pmatrix} \dot{x}_o / x_o^+ \\ \dot{x}_{\bar{o}} / x_{\bar{o}}^+ \end{pmatrix} = \begin{pmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{pmatrix} \begin{pmatrix} x_o \\ x_{\bar{o}} \end{pmatrix} + \begin{pmatrix} B_o \\ B_{\bar{o}} \end{pmatrix} u \qquad x_o \in \mathbb{R}^{\bar{n}}, \; x_{\bar{o}} \in \mathbb{R}^{n-\bar{n}}$$

$$y = \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{pmatrix} x_o \\ x_{\bar{o}} \end{pmatrix} + Du$$

Def 16.1 (A, C) is detectable if $n = \bar{n}$ or $A_{\bar{o}}$ is a stability matrix.

Basic idea: x_o is observable, and the unobservable states $x_{\bar{o}}$ converge to zero asymptotically.

16.4 Detectability Tests

All results follow from Duality to stabilizability.

Thm 16.4 (Eigenvector Test)

An ~~continuous time~~ LTI system is detectable iff every unstable eigenvector of A is not in the kernel of C

Recall: for continuous-time, unstable eigenvector is a eigenvector associated with a RHP eigenvalue

for discrete-time, an unstable eigenvector is an eigenvector associated with an eigenvalue on or outside the unit disk.

Thm 16.5 PBH test

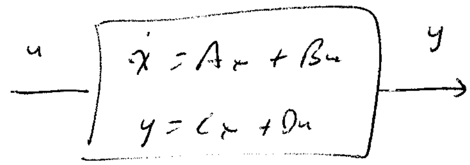
(A,C)- detectable iff rank $\left(\begin{bmatrix} A-\lambda I \\ \cancel{A}C \end{bmatrix}\right) = n$ $\forall$ unstable eigenvalue

Thm 16.6 (Lyapunov Test)

(A,C) - detectable iff $\exists$ a p.d. P s.t.

$$A^T P + PA - C^T C < 0 \quad / \quad A^T P A - P - C^T C < 0$$

16.5 State Estimation



Suppose that we would like to stabilize the system using the linear state feedback

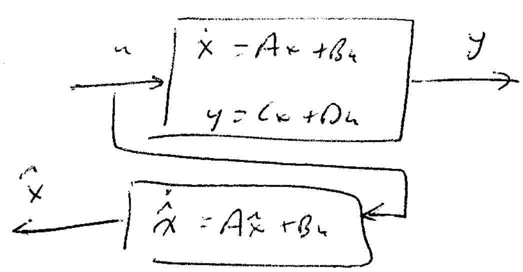
$$u = -Kx$$

so that $\dot{x} = (A - BK)x$

Unfortunately we don't measure x directly. However if the system is detectable, then it is possible to estimate $x(t)$ asymptotically.

Assuming (A, B, C, D) - known, we can construct the estimator

$$\dot{\hat{x}} = A\hat{x} + Bu \quad , \quad \hat{x}(0) = \hat{x}(0)$$



~~Define $e \triangleq \hat{x} - x$, then~~

$$\cancel{\dot{e} = \dot{\hat{x}} - \dot{x} = A\hat{x} + Bu - Ax - Bu = A(\hat{x} - x) = Ae}$$

$$\Rightarrow \cancel{e(t) = e^{At} e(0)}$$

Define $\tilde{x} \triangleq \hat{x} - x$, then

$$\dot{\tilde{x}} = \dot{\hat{x}} - \dot{x} = A\hat{x} + Bu - Ax - Bu = A(\hat{x} - x) = A\tilde{x} \quad , \quad \tilde{x}(0) = \hat{x}_0 - x_0$$

$$\Rightarrow \quad \tilde{x}(t) = e^{At}\tilde{x}_0$$

If eig(A) are in the LHP, then $\tilde{x}(t) \to 0$

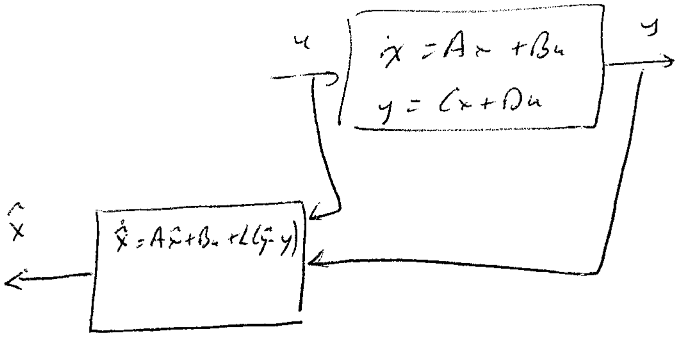
If not, then the estimation error diverges.

This naive estimator only uses the system inputs, and doesn't correct its estimate based on the output.

Alternatively, try

output injection gain

$$\dot{\hat{x}} = A\hat{x} + Bu - L\underbrace{(\hat{y} - y)}_{\text{innovation}}, \quad \hat{y} = C\hat{x} + Du$$



Again, let $\tilde{x} \triangleq \hat{x} - x$, then

$$\dot{\tilde{x}} = \dot{\hat{x}} - \dot{x} = A\hat{x} + Bu - L(C\hat{x} + Du - Cx - Du) - Ax - Bu$$

$$= (A - LC)(\hat{x} - x) = (A - LC)\tilde{x}$$

$$\Rightarrow \tilde{x}(t) = e^{(A-LC)t}\tilde{x}_0$$

<u>Thm 16.7</u> If the output injection gain $L \in \mathbb{R}^{n \times m}$ makes $(A - LC)$ a stability matrix, then $\tilde{x} \to 0$ for every input $u(t)$.

16.6 Eigenvalue assignment

Thm 16.8 If (A, C) is detectable, then it is possible to find L s.t. $A - LC$ is a stability matrix

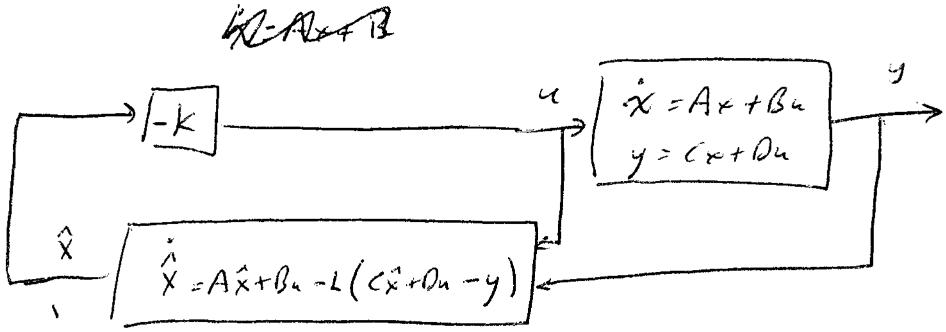
Thm 16.9 If (A, C) - observable, then the eigenvalues of $A - LC$ can be placed at arbitrarily defined locations.

Note, the easiest way to do this is using the Matlab command $L = \text{place}(A^T, C^T, [\lambda_1, \lambda_2, \ldots \lambda_n])$

The eigenvalues must appear in complex conjugate pairs.

16.7 Separation Principle

Suppose that we construct the following feedback system



Is the closed loop system stable ?

We can write the closed-loop system as

$$\dot{x} = Ax - BK\hat{x}$$
$$\dot{\hat{x}} = (A-LC)\hat{x} - BK\hat{x} + LCx$$

or

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A & -BK \\ +LC & A-LC-BK \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$$

Alternatively we can do a change of variables

$$\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$$

to get

$$\dot{x} = Ax - BK\tilde{x} - BKx = (A-BK)x - BK\tilde{x}$$
$$\dot{\tilde{x}} = \cancel{(A-LC)\hat{x} + (A-LC)x - BK\hat{x} - BKx - LCx}$$
$$\dot{\hat{x}} - \dot{x} = A\hat{x} - LC\hat{x} - BK\hat{x} - LCx - Ax + BK\hat{x}$$
$$= (A-LC)\tilde{x}$$

or

$$\begin{pmatrix} \dot{x} \\ \dot{\tilde{x}} \end{pmatrix} = \underbrace{\begin{pmatrix} A-BK & -BK \\ 0 & A-LC \end{pmatrix}} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}$$

$$eig = eig(A-BK) \cup eig(A-LC)$$

Therefore, in principle the state feedback K can be designed by placing $(A-BK)$ at arbitrary locations

and the output injection gain L can be designed separately by placing $(A-LC)$ at arbitrary locations.

In practice, this doesn't work very well, for reasons that will become clear later in the semester.

A simple explanation is as follows:

The closed loop eigenvalues are at

$$\mathrm{eig}(A-BK) \cup \mathrm{eig}(A-LC)$$

But what are the eigenvalues of the observer-based controller, which can be written as

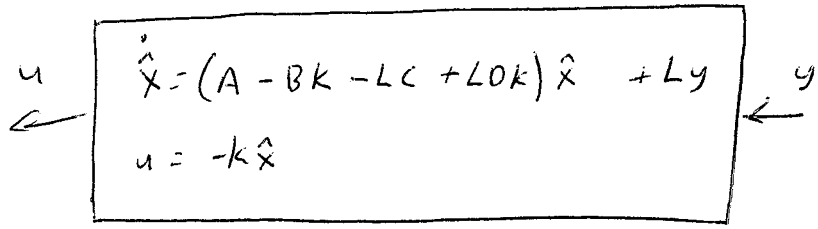
$$u \longleftarrow \boxed{\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu - L(\hat{y} - y) \\ u &= -K\hat{x} \end{aligned}} \longleftarrow y$$

or

$$\cancel{\dot{\hat{x}} = (A - BK - LC}$$

$$\cancel{\dot{\hat{x}} = A\hat{x} + BK}$$

$$u \longleftarrow \boxed{\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu - LC\hat{x} - LDu + Ly \\ u &= -K\hat{x} \end{aligned}} \longleftarrow y$$

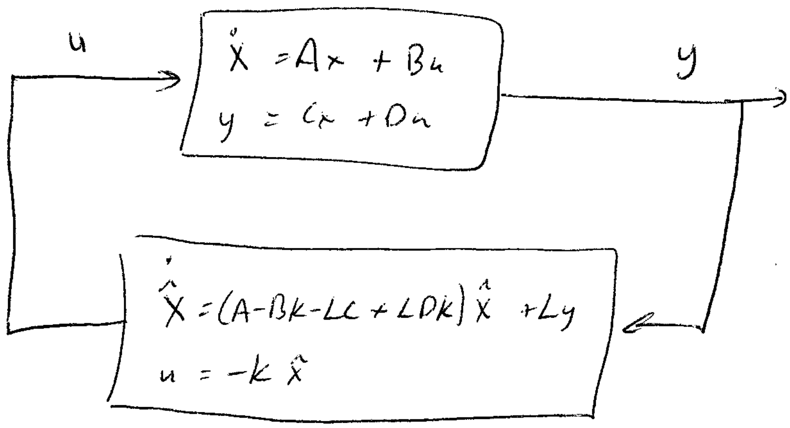


Note that this is a state space system with
"A-matrix" $= A - BK - LC + LDK$, "B-matrix" $= L$
"C-matrix" $= -K$ and "D-matrix" $= 0$

$\therefore$ The eigenvalues of the controller are

$$\text{eig}\,(A - BK - LC + LDK)$$

which ~~may be unstable~~ are not related to
$\text{eig}\,(A-BK) \cup \text{eig}\,(A-LC)$ and may be unstable!



open-loop poles : $\text{eig}\,(A) \cup \text{eig}\,(A - BK - LC + LDK)$

closed-loop poles : $\text{eig}\,(A - BK) \cup \text{eig}\,(A - LC)$

poles of the plant : $\text{eig}\,(A)$

poles of the controller : $\text{eig}\,(A - BK - LC + LDK)$