Lecture 18

Def: Two polynomials $a(s)$, $b(s)$ are co-prime if they have no common roots

Ex: $a(s) = (s+1)(s+2)$ $\quad$ $b(s) = (s+3)(s+4)(s+5)$

are co-prime

Ex: $a(s) = (s+1)(s+2)$ $\quad$ $b(s) = (s+2)(s+3)(s+4)$

are not co-prime.

For a SISO system:

$$g(s) = \frac{n(s)}{d(s)}$$

1. The poles of $g(s)$ are values of $s \in \mathbb{C}$ for which $g(s)$ becomes unbounded

2) The zeros of $g(s)$ are values of $s \in \mathbb{C}$ for which $g(s) = 0$

Note that for $g(s) = \dfrac{s+1}{(s+2)(s+3)}$

that there is a zero at infinity i.e. $|s| = \infty$.

True for all strictly proper transfer functions.

If $n(s)$ and $d(s)$ are co-prime, then the zeros are the roots of $n(s)$ (and possibly infinity)

and the poles are the roots of $d(s)$

In this chapter we extend these concepts of MIMO systems.

Informal definitions

1. The poles of $G(s)$ are the values of $s \in \mathbb{C}$ for which at least one entry becomes unbounded

2. The transmission zeros of $G(s)$ are those values of $s \in \mathbb{C}$ where $G(s)$ loses rank

18.2 Polynomial Matrices: Smith Form

Define $\mathbb{R}[s]^{m \times k}$ as the set of $m \times k$ matrices whose entries are polynomials in s with real coefficients.

$$P(s) = \begin{bmatrix} p_{11}(s) & \cdots & p_{1k}(s) \\ \vdots & & \\ p_{m1}(s) & \cdots & p_{mk}(s) \end{bmatrix} \in \mathbb{R}[s]^{m \times k}$$

~~The minors of $P(s)$~~

Example

$$P(s) = \begin{bmatrix} s & (s+1)(s+2) \\ (s+1) & (s+3) \end{bmatrix} \in \mathbb{R}[s]^{2 \times 2}$$

The minors of $P(s)$ of order i are the determinants of all square $i \times i$ submatrices of $P(s)$

Def. monic polynomial – leading coefficient is 1:

$$s^3 + 5s^2 + 4s + 6$$

Def monic greatest common divisor (gcd) of a family of polynomials is the monic polynomial of greatest order that divides all the polynomials in the family

Example

Find the gcd of

$\cancel{x^2+7x+6}$ $\quad$ s^2+7s+6

s^2-5x-6

Since $(s^2+7s+6) = (s+1)(s+6)$

$s^2-5x-6 = (s+1)(s-6)$

the gcd is $(s+1)$

Properties

1. If $p(s) = q(s) = 0$, then every polynomial is a common divisor and no gcd exists.

2. The constant 1 is always common divisor of $p(s), q(s)$.

   If $\gcd(p(s), q(s)) = 1$ then p and q are co-prime

3. $\gcd(p(s), q(s)) = \gcd(q(s), p(s))$

   $= \gcd(a_1 p(s) + b_1 q(s),\ a_2 p(s) + b_2 q(s))$

4. If $\gcd(p(s), r(s)) = 1$ then $\gcd(p(s), q(s)) = \gcd(p(s), q(s)r(s))$

5. $\gcd(p(s), q(s), r(s)) = \gcd(p(s), \gcd(q(s), r(s)))$

6. The gcd of two polynomials $p(s), q(s)$ is the smallest polynomial which can be written as a linear combination of $p(s), q(s)$, ie $\exists\, r(s), t(s)$ s.t.

$$d(s) = p(s)r(s) + q(s)t(s)$$

The determinantal divisors of $P(s)$ are polynomials

$$\{ D_i(s) \quad 0 \le i \le r \}$$

defined as follows

$D_0(s) = 1$

$D_i(s)$ - gcd of all nonzero minors of $P(s)$ of order i

Def The rank r of a $P(s)$ is the maximum order of a nonzero minor of $P(s)$

ie it is the first r for which the minors of $P(s)$ of order $r+1$ are all zero

Example 1

$$P(s) = \begin{pmatrix} s(s+2) & 0 \\ 0 & (s+1)^2 \\ (s+1)(s+2) & s+1 \\ 0 & s(s+1) \end{pmatrix}$$

order	Minors	Determinantal Divisor	invariant factors
$i=0$	none	$D_0(s) = 1$	
$i=1$	$s(s+2)$ $(s+1)^2$ $(s+1)(s+2)$ $(s+1)$ $s(s+1)$	$D_1(s) = 1$	$\varepsilon_1(s) = \frac{1}{1} = 1$
$i=2$	$s(s+1)^2(s+2)$ $s(s+1)(s+2)$ $s^2(s+1)(s+2)$ $-(s+1)^3(s+2)$ $s(s+1)^2(s+2)$	$D_2(s) = (s+1)(s+2)$	$\varepsilon_2(s) = (s+1)(s+2)$
$i=3$	not defined		

$\therefore$ rank $P(s) = 2$

Example 2

$$P(s) = \begin{pmatrix} s(s+2) & s(s+2)(s+1) \\ (s+1) & (s+1)^2 \end{pmatrix}$$

order	Minors	Determinantal Divisor	invariant factors
$i=0$	none	$D_0(s)=1$	
$i=1$	$s(s+2)$ $s(s+2)(s+1)$ $(s+1)$ $(s+1)^2$	$D_1(s)=1$	$\varepsilon_1(s)=1$ $\therefore$ rank $P(s)=1$
$i=2$	$s(s+2)(s+1)^2 - s(s+2)(s+1)^2 = 0$		

Def. The invariant factors of $P(s)$ are defined as

$$\varepsilon_i(s) = \frac{D_i(s)}{D_{i-1}(s)} \qquad i=1,\dots r$$

Lemma 18.1 For every $s_0 \in \mathbb{C}$

$$\text{rank } P(s_0) = \begin{cases} r & s_0 \text{ is not a root of } D_r(s) \\ <r & s_0 \text{ is a root of } D_r(s) \end{cases}$$

proof: At every $s_0 \in \mathbb{C}$ not a root of $D_r(s)$, one can find an $r \times r$ submatrix that is non singular $\Rightarrow r$ linearly indep rows & col.

At the roots of $D_r(s)$ every $r \times r$ submatrix is singular and has linearly dependent columns and rows.

Example

$$U(s) = \begin{pmatrix} 1 & s \\ 0 & 2 \end{pmatrix} \Rightarrow U^{-1}(s) = \frac{\begin{bmatrix} 2 & -s \\ 0 & 1 \end{bmatrix}}{2} = \begin{pmatrix} 1 & -s/2 \\ 0 & 1/2 \end{pmatrix}$$

$\therefore$ U is unimodular

$$W(s) = \begin{pmatrix} 1 & 1 \\ 0 & s \end{pmatrix} \Rightarrow W^{-1}(s) = \frac{\begin{bmatrix} s & -1 \\ 0 & 1 \end{bmatrix}}{s} = \begin{pmatrix} 1 & -1/s \\ 0 & 1/s \end{pmatrix}$$

$\therefore$ U is not unimodular.

Thm 18.1 For every real polynomial matrix $P(s) \in \mathbb{R}[s]^{m \times k}$ with Smith form $S_P(s)$, there exists unimodular real polynomial matrices $L(s) \in \mathbb{R}[s]^{m \times m}$, $R(s) \in \mathbb{R}[s]^{k \times k}$ s.t.

$$P(s) = L(s)\, S_P(s)\, R(s).$$

Definition The Smith form of $P(s) \in \mathbb{R}[s]^{m \times k}$ is ~~the~~

$$S_p(s) = \begin{bmatrix} \varepsilon_1(s) & & & & 0 \\ & \varepsilon_2(s) & & & \\ & & \ddots & & \\ 0 & & & \varepsilon_r(s) & 0 \\ & & 0 & & 0 \end{bmatrix} \in \mathbb{R}[s]^{m \times k}$$

Example 1 (cont)

$$S_p(s) = \begin{pmatrix} 1 & 0 \\ 0 & (s+1)(s+2) \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Example 2 (cont)

$$S_p(s) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Def: A square real polynomial matrix $U(s)$ is <u>unimodular</u> if its inverse is also a polynomial matrix.

Fact: A matrix is unimodular iff its determinant is a nonzero constant.

proof $$U^{-1} = \frac{\operatorname{adj}(U)}{\det(U)}$$

$\operatorname{adj}(U)$ is a polynomial matrix, but if $\det(U)$ is a polynomial in s, then U^{-1} will not be a polynomial matrix.

18.3 Rational Matrices : Smith-Mcmillan Form

Def: The set $\mathbb{R}(s)^{m \times k}$ denotes the set of matrices whose entries are ratios of polynomials with real coefficients

Def: The monic least common denominator (lcd) of a family of polynomials is the monic polynomial of smallest order that is divided by all polynomials in the family

Example:

$$p(s) = s(s+1)$$

$$q(s) = (s+1)(s+2)$$

$$\mathrm{lcd}(p(s), q(s)) = s(s+1)(s+2)$$

Any $G(s) \in \mathbb{R}(s)^{m \times k}$ can be written as

$$G(s) = \frac{1}{d(s)} N(s)$$

where $N(s) \in \mathbb{R}[s]^{m \times k}$ and $d(s)$ is the lcd of all ~~entries~~ the denominators of $G(s)$.

Def The Smith-McMillan form of $G(s) \in \mathbb{R}(s)^{m \times k}$ is

$$SM_G(s) = \frac{1}{d(s)} S_N(s) = \begin{bmatrix} \frac{\eta_1(s)}{\psi_1(s)} & & & 0 \\ & \ddots & & \\ & & \frac{\eta_r(s)}{\psi_r(s)} & \\ & & & \\ 0 & & & 0 \end{bmatrix} \in \mathbb{R}(s)^{m \times k}$$

where $S_N(s) \in \mathbb{R}[s]^{m \times k}$ is the smith form of $N(s)$

and where all pairs $\{\eta_i(s), \psi_i(s)\}$ are co-prime

Note: Since each $\varepsilon_{i-1}(s)$ divides $\varepsilon_i(s)$,

therefore each $\psi_i(s)$ divides $\psi_{i-1}(s)$

Why??

<u>Also</u>: $\frac{\eta_i(s)}{\psi_i(s)}$ are not necessarily proper

<u>Thm 18.2 Smith - McMillan Factorization</u>

for every $G(s) \in \mathbb{R}(s)^{m\times k}$, $\exists$ unimodular $L(s) \in \mathbb{R}[s]^{m\times m}$

and $R(s) \in \mathbb{R}[s]^{k\times k}$ s.t.

$$G(s) = \frac{1}{d(s)} N(s) = L(s)\, SM_G(s)\, R(s)$$

<u>18.4 McMillan Degree, poles, zeros</u>

<u>Def 18.3</u> For $G(s) \in \mathbb{R}(s)^{m\times k}$ in Smith-McMillan form,

the polynomial
$$P_G(s) = \psi_1(s)\psi_2(s)\cdots\psi_r(s)$$
is called the <u>pole polynomial</u> of $G(s)$.

Its degree is the <u>McMillan degree</u>.

Its roots are the poles of $G(s)$.

The polynomial $Z_G(s) = \eta_1(s)\cdots\eta_r(s)$

is called the <u>zero polynomial</u> of $G(s)$.

Its roots are called the <u>transmission zeros</u> of $G(s)$

Example

$$\mathcal{G}(s) = \begin{bmatrix} \frac{s+2}{s+1} & 0 \\ 0 & \frac{s+1}{s} \\ \frac{s+2}{s} & \frac{1}{s} \\ 0 & 1 \end{bmatrix} = \frac{1}{s(s+1)} N(s) \quad \text{where } N(s) = \begin{bmatrix} s(s+2) & 0 \\ 0 & (s+1)^2 \\ (s+1)(s+2) & s+1 \\ 0 & s(s+1) \end{bmatrix}$$

We already computed the Smith form of $N(s)$ as

$$S_N(s) = \begin{bmatrix} 1 & 0 \\ 0 & (s+1)(s+2) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \quad SM_{\mathcal{G}}(s) = \frac{1}{d(s)} S_N(s) = \begin{bmatrix} \frac{1}{s(s+1)} & 0 \\ 0 & \frac{s+2}{s} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The pole polynomial is

$$p_{\mathcal{G}}(s) = s^2(s+1)$$

The McMillan degree is 3, poles are $\{0, 0, -1\}$

The zero polynomial is

$$Z_{\mathcal{G}}(s) = s+2$$

The transmission zero is at $\{-2\}$

For scalar transfer functions

$$g(s) = k \frac{n(s)}{d(s)}$$

If $(n(s), d(s))$ are co-prime ~~then~~ and monic,

then the Smith-McMillan form is

$$SM_g(s) = \frac{n(s)}{d(s)}$$

The zeros are the roots of $n(s)$, the poles are the roots of $d(s)$.

For square $G(s) \in \mathbb{R}(s)^{m \times m}$ with $r = m$ (full rank)

then

$$\det G(s) = \det\left(L(s)\, SM_G(s)\, R(s)\right)$$

$$= \det L(s) \det R(s) \det SM_G(s)$$

$$= k \frac{Z_G(s)}{P_G(s)} \quad \text{where } k = \det L(s) \det R(s) \equiv \text{constant.}$$

However if there are common poles and zeros (possible for MIMO systems) then they do not show up in $\det G(s)$!

Example:

$$G(s) = \begin{bmatrix} \frac{1}{s(s-2)} & 0 \\ 0 & \frac{s-2}{s} \end{bmatrix}$$

Note $G(s)$ is already in Smith-McMillan form,

The ~~McM~~ pole polynomial is $P_G(s) = s^2(s-2)$, and

the zero polynomial is $Z_G(s) = s-2$.

McMillan degree $= 3$

poles $= \{0, 0, 2\}$

zeros $= \{2\}$

but $\det G(s) = \frac{1}{s^2}$

(ie the pole & zero at 2 are hidden)

18.5 Transmission Zeros and Transmission Blocking

Given $\dot{x} = Ax + Bu$, $y = Cx + Du$

with rational transfer matrix

$$\hat{G}(s) = C(sI-A)^{-1}B + D$$