Lecture 19

Given $\dot{x}/x^+ = Ax + Bu$, $y = Cx + Du$

$$\hat{G}(s) = C(sI - A)^{-1}B + D \in \mathbb{R}(s)^{m \times k}$$

Thm 19.1

$$\{\text{poles of } \hat{G}(s)\} \subset \{\text{eigenvalues of } A\}$$

proof:

$$\hat{G}(s) = \frac{1}{\Delta(s)}\left[C \,\text{adj}(sI - A) B + \Delta(s) D\right]$$

where $\Delta(s) = \det(sI - A)$

$$= \frac{1}{d(s)} N(s)$$

where $d(s)$ – monic least common denominator of ~~the~~ element of $\hat{G}(s)$

and $N(s) \in \mathbb{R}[s]^{m \times k}$

$\therefore$ $d(s)$ equals $\Delta(s)$ except for cancellations

$$\Rightarrow \{\text{roots of } d(s)\} \subset \{\text{roots of } \Delta(s)\} \in \{\text{e.v. of } A\}$$

The Smith-McMillan form is

$$SM_{\hat{G}}(s) = \begin{bmatrix} \frac{\eta_1(s)}{\psi_1(s)} & & & \\ & \ddots & & 0 \\ & & \frac{\eta_r(s)}{\psi_r(s)} & \\ 0 & & & 0 \end{bmatrix} = \frac{1}{d(s)} SN(s)$$

where $P_{\hat{G}}(s) = \prod \psi_i(s)$ ~~[illegible]~~ roots of $P_{\hat{G}}(s)$ are also roots of $d(s)$

$$\Rightarrow \{\text{poles of } \hat{G}(s)\} = \{\text{roots of } P_{\hat{G}}(s)\} \in \{\text{roots of } d(s)\} \subset \{\text{e.v. of } A\}$$

19.2 Transmission zeros vs Invariant zeros

Given the state-space realization

$$\dot{x} = Ax + Bu \quad , \quad y = Cx + Du \qquad (2.1)$$

Take Laplace transform

$$s\hat{x}(s) - \hat{x}(0) = A\hat{x}(s) + B\hat{u}(s) \quad , \quad \hat{y}(s) = C\hat{x}(s) + D\hat{u}(s)$$

rewrite as

$$\begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} -\hat{x}(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} -x(0) \\ \hat{y}(s) \end{bmatrix}$$

$$P(s) \triangleq \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix}$$ – Rosenbrock's system matrix

Def 19.1 (Invariant zeros)

The invariant zero polynomial of (2.1) is the monic greatest common divisor $z_P(s)$ of all nonzero minors of order $r = \text{rank}\, P(s)$. The roots of $z_P(s)$ are called the ~~zeros~~ invariant zeros of (2.1)

Example 19.1

$$\dot{x} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x$$

$$P(s) = \begin{bmatrix} s & 1 & -1 & 1 & 0 \\ -1 & s+2 & -1 & 1 & 1 \\ 0 & -1 & s+1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{rank is at most } 4$$

$$\begin{vmatrix} s & 1 & -1 & 1 \\ -1 & s+2 & -1 & 1 \\ 0 & -1 & s+1 & 1 \\ 0 & 1 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} s & -1 & 1 \\ -1 & -1 & 1 \\ 0 & s+1 & 1 \end{vmatrix} = s\begin{vmatrix} -1 & 1 \\ s+1 & 1 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ s+1 & 1 \end{vmatrix} = s(-1-s-1) + 1(-1-s-1)$$
$$= -(s+1)(s+2)$$

$$\begin{vmatrix} s & 1 & -1 & 0 \\ -1 & s+2 & -1 & 1 \\ 0 & -1 & s+1 & 2 \\ 0 & 1 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} s & -1 & 0 \\ -1 & -1 & 1 \\ 0 & s+1 & 2 \end{vmatrix} = s\begin{vmatrix} -1 & 1 \\ s+1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & 0 \\ s+1 & 2 \end{vmatrix} = s(-2-s-1) + 1(-2)$$
$$= -s^2-3s-2$$
$$= -(s+1)(s+2)$$

$$\begin{vmatrix} s & 1 & 1 & 0 \\ -1 & s+2 & 1 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 1 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} s & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = s\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = s+2$$

$$\begin{vmatrix} s & -1 & 1 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & s+1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & -1 & 1 & 0 \\ s+2 & -1 & 1 & 1 \\ -1 & s+1 & 1 & 2 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 & 0 \\ -1 & 1 & 1 \\ s+1 & 1 & 2 \end{vmatrix} = -1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ s+1 & 2 \end{vmatrix} = -1 - (-2-s-1)$$
$$= s+2$$

$\therefore \quad Z_p(s) = s+2$

$\Rightarrow$ single invariant zero at $\{-2\}$

<u>Blocking property</u>

If z_0 is an invariant zero of $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$, then

$\exists \begin{pmatrix} -x_0 \\ u_0 \end{pmatrix} \in \ker P(z_0) \Rightarrow \begin{pmatrix} z_0 I - A & B \\ -C & D \end{pmatrix}\begin{pmatrix} -x_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \quad z_0 x_0 = A x_0 + B u_0$

$C x_0 + D u_0 = 0$

~~Pf~~ Let ~~then~~ $x(t) = e^{z_0 t} x_0$ and $u(t) = e^{z_0 t} u_0$

then $\dot{x}(t) = z_0 e^{z_0 t} x_0$

$$
\begin{aligned}
Ax + Bu &= e^{z_0 t}(Ax_0 + Bu_0) \\
&= z_0 e^{z_0 t} x_0 \\
&= \dot{x}(t)
\end{aligned}
$$

and $y(t) = Cx(t) + Du(t) = e^{z_0 t}(Cx_0 + Du_0) = 0$

<u>Property 19.1</u> For every invariant zero z_0 of (2.1) that is not an eigenvalue of A, $\exists$ a nonzero input of the form $u(t) = e^{z_0 t} u_0$ and ~~the~~ initial state x_0 for which the output is identically zero.

<u>Thm 19.2</u>

$$\{\text{transmission zeros of } \hat{G}(s)\} \subset \{\text{invariant zeros of (LTI)}\}$$

<u>proof</u>:

$$
P(s) = \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix} = \begin{bmatrix} sI - A & 0 \\ -C & I \end{bmatrix} \begin{bmatrix} I & (sI-A)^{-1}B \\ 0 & \underbrace{C(sI-A)^{-1}B + D}_{\hat{G}(s)} \end{bmatrix}
$$

so if z_0 is a transmission zero of $\hat{G}(s)$ then since $\hat{G}(z_0)$ drops in rank, so must $P(z_0)$.

19.3 Order of Minimal Realization

For $\dot{x}/x^+ = Ax + Bu, \quad y = Cx + Du$ (5.1)

$\hat{G}(s) = C(sI - A)^{-1}B + D$

we know that

$$\{\text{poles of } \hat{G}(s)\} \subset \{\text{eigs of } A\}$$

$$\{\text{transmission zeros of } \hat{G}(s)\} \subset \{\text{invariant zeros of } (5.1)\}$$

Thm 19.3 The realization (5.1) is minimal iff $n =$ ~~dim~~ size(A) is equal to the McMillan degree of $\hat{G}(s)$. In which case

$$\{\text{poles of } \hat{G}(s)\} = \{\text{eigs of } A\}$$

$$\{\text{transmission zeros of } \hat{G}(s)\} = \{\text{invariant zeros of } (5.1)\}$$

Corollary 19.1 If realization (5.1) of $\hat{G}(s)$ is minimal, then $\hat{G}(s)$ is BIBO stable iff (5.1) is asymptotically stable in the sense of Lyapunov.

Example 19.2

$$\dot{x} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} u, \quad y = [0 \ 1 \ 0]\, x$$

$\{\text{eig of } A\} = \{0, -1, -2\}$ and we saw that the invariant zero is at -2.

$$\hat{g}(s) = [0 \ 1 \ 0] \begin{pmatrix} s & 1 & -1 \\ -1 & s+2 & -1 \\ 0 & -1 & s+1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} = \frac{1}{s}[1 \ 1]$$

$$S_N(s) = [1 \ 0] \quad \Rightarrow \quad SM_{\hat{g}(s)}(s) = [\tfrac{1}{s} \ 0]$$

$\Rightarrow$ no transmission zeros, single pole at $s=0$

Kalman decomposition

$$\dot{\bar{x}} = \left(\begin{array}{cc|c} 0 & 0 & -\sqrt{2} \\ 0 & -1 & -\sqrt{3} \\ \hline 0 & 0 & -2 \end{array}\right) \bar{x} + \left(\begin{array}{cc} \sqrt{3} & \sqrt{3} \\ 0 & -\sqrt{2} \\ \hline 0 & 0 \end{array}\right) u,$$

A_{co}, $A_{c\bar{o}}$, $A_{\bar{c}\bar{o}}$, B_{co}, $B_{c\bar{o}}$

$$y = \left(\begin{array}{cc|c} \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{2}}{3} \end{array}\right) x$$

C_{co} $C_{\bar{o}}$ $C_{\bar{c}o}$

Only one controllable and observable mode.

19.7 System Inverse

Given the system

$$u_{(k\times 1)} \longrightarrow \boxed{\begin{array}{l} \underbrace{\dot{x}/x^+}_{n\times 1} = Ax + Bu \\ \underbrace{y}_{m\times 1} = Cx + D\underbrace{u}_{k\times 1} \end{array}} \longrightarrow y_{(m\times 1)} \qquad (7.1)$$

We say that

$$\bar{u}_{(m\times 1)} \longrightarrow \boxed{\begin{array}{l} \underbrace{\dot{\bar{x}}/\bar{x}^+}_{\bar{n}\times 1} = \bar{A}\bar{x} + \bar{B}\bar{u} \\ \underbrace{\bar{y}}_{k\times 1} = \bar{C}\bar{x} + \bar{D}\underbrace{\bar{u}}_{m\times 1} \end{array}} \longrightarrow \bar{y}_{(k\times 1)} \qquad (7.2)$$

is an inverse of (7.1) if ~~the cascade~~ for the following interconnections:

$$u \longrightarrow \boxed{\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array}} \xrightarrow{y = \bar{u}} \boxed{\begin{array}{l} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u} \\ \bar{y} = \bar{C}\bar{x} + \bar{D}\bar{u} \end{array}} \longrightarrow \bar{y} \qquad (7.3)$$

$$\bar{u} \longrightarrow \boxed{\begin{array}{l} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u} \\ \bar{y} = \bar{C}\bar{x} + \bar{D}\bar{u} \end{array}} \xrightarrow{\bar{y} = u} \boxed{\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array}} \longrightarrow y \qquad (7.4)$$

where $x(0) = 0$ and $\bar{x}(0) = 0$ we have $u = \bar{y}$ for (7.3)

and $\bar{u} = y$ for (7.4)

In terms of transfer functions

(7.3) gives $\quad \hat{\bar{G}}(s)\hat{G}(s) = I \quad$ (left inverse)

(7.4) gives $\quad \hat{G}(s)\hat{\bar{G}}(s) = I \quad$ (right inverse)

if $(m = k)$ square then $\hat{\bar{G}}(s) = \hat{G}^{-1}(s)$

Note that the input and output of the cascade will be identical only for zero initial conditions.

If the initial conditions are nonzero **and** $\hat{G}(s)$ and $\hat{G}^{-1}(s)$ are stable then, for any input

$$\lim_{t \to \infty} \| u(t) - \bar{y}(t) \| = 0$$

or

$$\lim_{t \to \infty} \| \bar{u}(t) - y(t) \| = 0$$

19.5 Existence of Inverse

$$\dot{x}/x^+ = Ax + Bu \;, \qquad y = Cx + Du \qquad (LTI)$$

Assume that $k = m$ and D is nonsingular

then $$u = -D^{-1}Cx + D^{-1}y$$

$$\Rightarrow \quad \dot{x}/x^+ = Ax + B(-D^{-1}Cx + D^{-1}y)$$
$$= (A - BD^{-1}C)x + BD^{-1}y$$

$$\Rightarrow \quad \dot{\bar{x}}/\bar{x}^+ = (A - BD^{-1}C)\bar{x} + BD^{-1}\bar{u} \;, \quad \bar{y} = -D^{-1}C\bar{x} + D^{-1}\bar{u}$$

is an inverse of (LTI).

Thm 19.4 The system (LTI) has an inverse iff D is nonsingular. Moreover $\hat{G}^{-1}(s) = \hat{\bar{G}}(s)$

where $$\hat{G}(s) = C(sI-A)^{-1}B + D$$
$$\hat{\bar{G}}(s) = \bar{C}(sI-\bar{A})^{-1}\bar{B} + \bar{D}$$

and $\bar{A} = A - BD^{-1}C$, $\bar{B} = BD^{-1}$, $\bar{C} = -D^{-1}C$, $\bar{D} = D^{-1}$

19.6 Poles and Zeros of Inverse

for $\hat{G}(s) \in \mathbb{R}(s)^{m \times k}$ with Smith-McMillan form

$$SM_G(s) = \left[\begin{array}{ccc|c} \frac{\eta_1(s)}{\psi_1(s)} & & 0 & \\ & \ddots & & \\ 0 & & \frac{\eta_r(s)}{\psi_r(s)} & 0 \\ \hline & 0 & & 0 \end{array}\right]$$

there exists unimodular real polynomial matrices $L(s)$, $R(s)$ s.t.

$$\hat{G}(s) = L(s)\, SM_G(s)\, R(s)$$

Since $L(s)$ and $R(s)$ have constant non-zero determinants, $\hat{G}(s)$ has an inverse iff $SM_G(s)$ is invertible

$\Rightarrow m = r$

$\Rightarrow$
$$\hat{G}^{-1}(s) = R^{-1}(s)\, SM_G^{-1}(s)\, L^{-1}(s)$$

where
$$SM_G^{-1}(s) = \begin{pmatrix} \frac{\psi_1(s)}{\eta_1(s)} & & 0 \\ & \ddots & \\ 0 & & \frac{\psi_r(s)}{\eta_r(s)} \end{pmatrix}$$

Properties 19.2

1. The poles of $\hat{G}(s)^{-1}$ are the transmission zeros of $\hat{G}(s)$ and vise versa
2. $\hat{G}(s)^{-1}$ is BIBO stable iff every transmission zero of $\hat{G}(s)$ has strictly negative real part

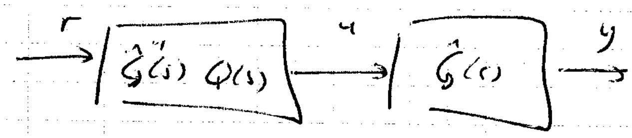
Def: When the transmission zeros of $\hat{G}(s)$ have strictly neg real part, the system is called (strictly) minimum phase.

Facts for square invertible systems

1. If $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a minimal realization of $\hat{G}(s)$ then
$$\begin{bmatrix} A-BD^{-1}C & BD^{-1} \\ -D^{-1}C & D^{-1} \end{bmatrix}$$ is a minimal realization of $\hat{G}^{-1}(s)$.
2. ~~McMill~~ Invertible systems with McMillan degree n, have exactly n transmission zeros.
3. The transmission zeros of minimal realization $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ are $\text{eig}(A-BD^{-1}C)$.

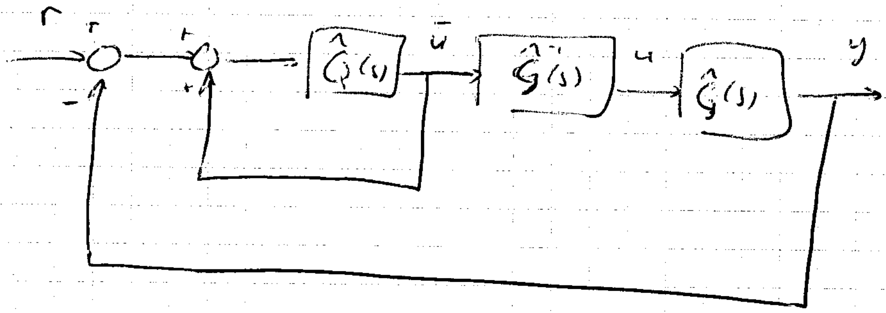
19.7 Feedback Control of Invertible Systems with Stable inverses

Open-loop control



$$Y(s) = \hat{G}(s)\, U(s) = \hat{G}(s)\, \hat{G}^{-1}(s)\, Q(s)\, R(s) = Q(s)\, R(s)$$

Closed-loop control



$$Y = \hat{G}\, \hat{G}^{-1}\, \bar{u}$$

$$\bar{u} = \hat{Q}\,(R - Y + \bar{u}) \quad \Rightarrow \quad (I - Q)\,\bar{u} = \hat{Q}\,(R - Y)$$

$$\Rightarrow \quad \bar{u} = (I - \hat{Q})^{-1}\, \hat{Q}\,(R - Y)$$

So

$$Y = \hat{G}\, \hat{G}^{-1} (I - \hat{Q})^{-1} \hat{Q}\,(R - Y) = (I - \hat{Q})^{-1} \hat{Q} R - (I - \hat{Q})^{-1} \hat{Q}\, Y$$

$$\Rightarrow \quad \left[I + (I - \hat{Q})^{-1} \hat{Q}\right] Y = [I - Q]^{-1}\, \hat{Q} R$$

$$\Rightarrow \quad Y = \left[I + (I - \hat{Q})^{-1} \hat{Q}\right]^{-1} (I - \hat{Q})^{-1}\, \hat{Q} R$$

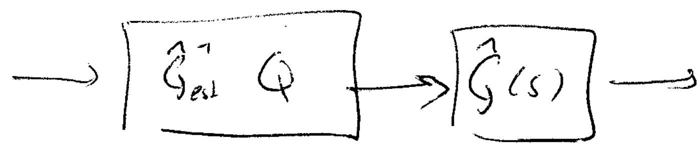
$$= \left[(I - \hat{Q}) + (I - Q)(I - Q)^{-1} \hat{Q}\right]^{-1} \hat{Q} R$$

$$= \left[I - \hat{Q} + \hat{Q}\right] \hat{Q} R$$

$$= \hat{Q} R$$

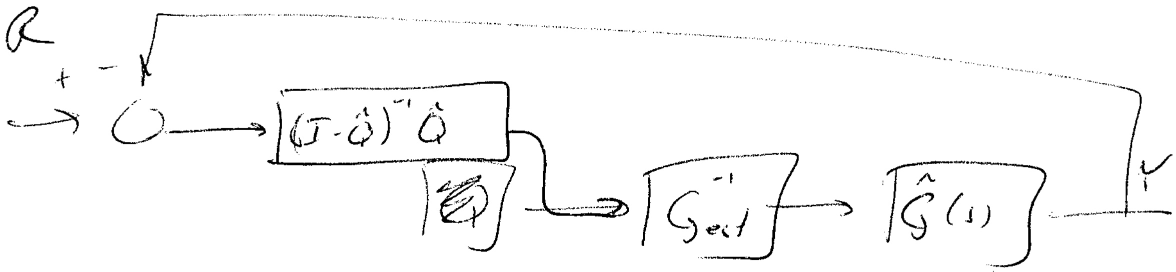
Same transfer function!

o open loop



$$Y = \underbrace{\hat{G}\,\hat{G}_{est}^{-1}}_{\text{direct degradation}} Q R$$

R



$$\bar{u} = \hat{Q}(s + \bar{u}) = \hat{Q}s + \hat{Q}\bar{u}$$

$$(I - \hat{Q})\bar{u} = \hat{Q}s$$

$$\bar{u} = (I - \hat{Q})^{-1}\hat{Q}\,s \qquad \lim_{s\to 0} Q(s) = K$$

$$Y = \hat{G}\,\hat{G}_{est}^{-1}(I - \hat{Q})^{-1}\hat{Q}(R - Y)$$

$$[I - G G_{est}^{-1}(I - Q)^{-1} Q]\,Y = \hat{G} G_{est}^{-1}(I - \hat{Q})^{-1} Q R$$

$$Y = [I - G G_{est}^{-1}(I - Q)^{-1} Q]^{-1} G G_{est}^{-1}(I - \hat{Q})^{-1} Q R$$

$$= [(I - \hat{Q}) G_{est} G^{-1} - (I - Q) G_{est} G^{-1} G G_{est}^{-1}(I - Q)\,Q]^{-1} Q R$$

$$= [(I - Q) G_{est} G^{-1} - Q]^{-1} Q R$$

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} s\,[(I - K)M - K]^{-1} K R$$

$Y = \left[G_{est}\, G^{-1} - (I-Q)^{-1} Q \quad \right]^{-1} (I-Q)^{-1} Q R$

$(I-Q)^{-1} Q$ $\qquad q = \frac{k}{s+k}$

$(I-Q) = \left(\frac{s+k-k}{s+k} \right)^{-1} \left(\frac{k}{s+k} \right) = \left(\frac{s+k}{s} \right) \left(\frac{k}{s+k} \right) = \frac{k}{s}$

$\lim_{s \to 0}$

$Y =$

ok if $\lim_{s \to \infty} (I-Q)^{-1} Q = \infty$

and $\lim_{s \to 0} \frac{G_{est}(s)\, G^{-1}(s)}{g} =$ constant