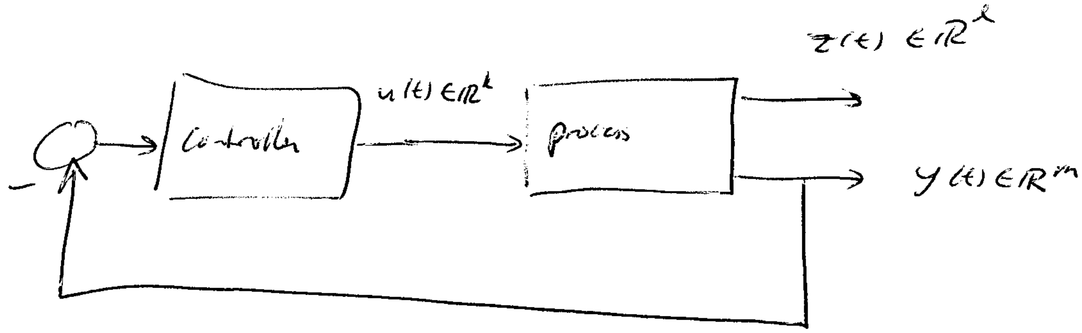
Lecture 20 LQR



process :

$$\dot{x} = Ax + Bu$$ ← state equation

$$y = Cx$$ ← ~~output or~~ measured output (from sensors)

$$z = Gx + Hu$$ ← controlled output – things you want to make small

Example : to make y small let

$$G = C \quad \text{and} \quad H = 0$$

$$\Rightarrow z = Cx = y$$

Example : to make $\begin{pmatrix} y \\ \dot{y} \end{pmatrix}$ small, let

$$z = \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} Cx \\ C\dot{x} \end{pmatrix} = \begin{pmatrix} Cx \\ CAx + CBu \end{pmatrix} = \underbrace{\begin{pmatrix} C \\ CA \end{pmatrix}}_{G} x + \underbrace{\begin{pmatrix} 0 \\ CB \end{pmatrix}}_{H} u$$

20.2 Optimal Regulation

LQR optimality criteria:

$$J_{LQR} = \int_0^\infty \underbrace{\|z(t)\|^2}_{\substack{\text{energy of} \\ \text{controlled} \\ \text{output}}} + \underbrace{\rho\|u(t)\|^2}_{\substack{\text{energy of} \\ \text{controlled} \\ \text{input.}}} dt \qquad \rho > 0$$

ρ - large implies $\int_0^\infty \|u(t)\| dt$ small ~~in comparison to~~ at the expense of $z(t)$

ρ - small - $z(t)$ small at the expense of large input

More generally we have

$$J_{LQR} = \int_0^\infty z^T \bar{Q} z + \rho u^T \bar{R} u \, dt$$

Since $z = Gx + Hu$ we get

$$J_{LQR} = \int_0^\infty (Gx + Hu)^T \bar{Q} (Gx + Hu) + \rho u^T \bar{R} u \quad dt$$

$$= \int_0^\infty x^T G^T \bar{Q} G x + u^T(\rho\bar{R} + H^T \bar{Q} H) u + 2x^T G^T \bar{Q} H u \, dt$$

letting $Q \hat{=} G^T \bar{Q} G$, $R \hat{=} \rho\bar{R} + H^T \bar{Q} H$, $N \hat{=} G^T \bar{Q} H$

$$J_{LQR} = \int_0^\infty x^T Q x + u^T R u + 2x^T N u \, dt$$

20.3 Feedback Invariants

Recall: Given $\dot{x} = Ax + Bu \qquad x(0) = x_0$ (3.1)

$H(x(\cdot), u(\cdot))$ is a feedback invariant if it only depends on x_0 and not $u(t),\ t \geq 0$

Proposition 20.1 If $\lim_{t \to \infty} x(t) = 0$ then for every symmetric P,

$$H(x(\cdot), u(\cdot)) = -\int_0^\infty (Ax + Bu)^T P x + x^T P (Ax + Bu)\, dt \qquad (3.2)$$

is a feedback invariant

proof: $$H(x(\cdot), u(\cdot)) = -\int_0^\infty \frac{d}{dt}\left(x^T(t) P x(t)\right) dt$$

$$= -\lim_{t \to \infty} x^T(t) P x(t) + x_0^T P x_0$$

$$= x_0^T P x_0$$

20.4 If

$$J = H(x(\cdot), u(\cdot)) + \int_0^\infty \Lambda(x(t), u(t))\, dt$$

where H is a feedback invariant and $\Lambda(x, u)$ satisfies

$$\min_{w \in \mathbb{R}^k} \Lambda(x, w) = 0$$

then the control that minimizes J is

$$u^*(t) = \arg\min_{w \in \mathbb{R}^k} \Lambda(x(t), w)$$

and $J_*^* = H(x(\cdot), u(\cdot))$

20.5 Optimal State Feedback

$$J_{LQR} = \int_0^\infty x^T Q x + u^T R u + 2 x^T N u \; dt$$

The idea is to add and subtract the feedback invariant (8.2)

$$J_{LQR} = H(x(\cdot), u(\cdot)) + \int_0^\infty \Big[x^T Q x + u^T R u + 2 x^T N u + (Ax+Bu)^T P x + x^T P (Ax + Bu) \Big] dt$$

$$= H(x(\cdot), u(\cdot)) + \int_0^\infty \Big[x^T (A^T P + PA + Q) x + \underbrace{u^T R u + 2 u^T (B^T P + N^T) x}_{\text{complete the square}} \Big] dt$$

$$(u + kx)^T R (u + kx) = u^T R u + u^T R K x + x^T K^T R u + x^T K^T R K x$$
$$= u^T R u + 2 u^T R K x + x^T K R K x$$

comparing terms, let $RK = B^T P + N^T$

$$\Rightarrow K = R^{-1}(B^T P + N^T)$$

$\therefore$

$$J_{LQR} = H(x(\cdot), u(\cdot)) + \int_0^\infty \Big[(u + Kx)^T R (u + Kx) + x^T (A^T P + PA + Q - K^T R K)\, x \Big]\, dt \qquad (5.1)$$

Since $H(x(\cdot), u(\cdot))$ is a feedback invariant for any symmetric P, ~~select~~ select P as the symmetric solution of

$$A^T P + PA + Q - K^T R K = 0 \qquad (5.2)$$

to get

$$J_{LQR} = H(x(\cdot), u(\cdot)) + \int_0^\infty (u + Kx)^T R (u + Kx)\, dt$$

Note that if $R > 0$ then

$$\min_{w \in \mathbb{R}^k} (w + Kx)^T R (w + Kx) = 0$$

and

$$u^*(t) = \arg\min_{w \in \mathbb{R}^k} (w + Kx)^T R (w + Kx)$$

$$= -K x(t)$$

or

$$\boxed{u^*(t) = R^{-1} (B^T P + N^T)\, x(t)} \qquad (5.3)$$

Eq (5.2) becomes

$$A^T P + PA + Q - \left(R^{-1} (B^T P + N^T) \right)^T R \left(R^{-1} (B^T P + N^T) \right) = 0$$

$$\Longleftrightarrow \boxed{A^T P + PA + Q - (PB + N) R^{-1} (B^T P + N^T) = 0} \qquad (5.4)$$

Algebraic Riccati Equation

Discrete time

$$x_{k+1} = A x_k + B u_k \qquad \text{the } x_0 \text{ - initial condition}$$

$$y_k = C x_k$$

$$z_k = G x_k + H u_k$$

$$J_{LQR} = \sum_{j=0}^{\infty} x_j^T Q x_j + u_j^T R u_j + 2 x_j^T N u_j$$

For discrete-time systems, we need a feedback invariant

try

$$H(x(\cdot), u(\cdot)) = -\sum_{j=0}^{\infty} \left\{ (A x_j + B u_j)^T P (A x_j + B u_j) - x_j P x_j \right\}$$

$$= -\sum_{j=0}^{\infty} \left\{ x_{j+1}^T P x_{j+1} - x_j^T P x_j \right\}$$

$$= -\left\{ \begin{array}{l} (x_1^T P x_1 - x_0^T P x_0) \\ + (x_2^T P x_2 - x_1^T P x_1) \\ + (x_3^T P x_3 - x_2^T P x_2) \\ + \cdots \end{array} \right\}$$

$$= x_0^T P x_0 - \lim_{j \to \infty} x_j^T P x_j$$

$\therefore$ If $x_j \to 0$ then $H(x(\cdot), u(\cdot))$ is a feedback invariant for any P

~~The~~ Next step is to write J_{LQR} ~~rates~~ in terms of H:

$$J_{LQR} = H(x(\cdot),u(\cdot)) + \sum_{j=0}^{\infty}\Big\{x_j^T Q x_j + u_j^T R u_j + 2x_j^T N u_j - x_j^T A^T P A x_j - u_j^T B^T P B u_j - 2u_j^T B^T P A x_j - x_j^T P x_j\Big\}$$

$$= H(x(\cdot),u(\cdot)) + \sum_{j=0}^{\infty}\Big\{u_j^T(R - B^T P B)u_j + 2u_j^T(B^T P A - N^T)x_j - x_j^T(A^T P A + P - Q)x_j\Big\}$$

complete square

$$(u_j + \hat{k}x_j)^T \bar{R}(u_j + \hat{k}x_j) = u_j^T \bar{R} u_j + 2u_j^T \bar{R}\hat{k}x_j + x_j^T \hat{k}^T \bar{R}\hat{k}x_j$$

comparing terms, let $\bar{R} = (R - B^T P B)$

and $\bar{R}\hat{k} = B^T P A - N^T$

$$\Rightarrow \boxed{K = (R - B^T P B)^{-1}(B^T P A - N^T)}$$

$$\Rightarrow J_{LQR} = H(x(\cdot),u(\cdot)) + \sum_{j=0}^{\infty}\Big\{(u_j + Kx_j)^T(R - B^T P B)(u_j + Kx_j) - x_j^T\big(A^T P A + P - Q - K^T(R - B^T P B)K\big)x_j\Big\}$$

If we select P as the solution of

$$\boxed{A^T P A + P - Q - (A^T P B - N)(R - B^T P B)^{-1}(B^T P A - N^T) = 0}$$

then

$$J_{LQR} = H(x(\cdot),u(\cdot)) + \sum_{j=0}^{\infty}(u_j + Kx_j)^T(R - B^T P B)(u + Kx_j)$$

if $R - B^T P B > 0$ then $\boxed{u_j^* = -Kx_j}$

Bryson's Rule

For

$$J_{LQR} = \int_0^\infty \left(z^T \bar{Q} z + \rho u^T \bar{R} u \right) dt$$

Bryson's rule is to select $\bar{Q}$ and $\bar{R}$ as diagonal matrices with

$$\bar{Q}_{ii} = \frac{1}{\text{max acceptable value of } z_i^2}$$

$$\bar{R}_{ii} = \frac{1}{\text{max acceptable value of } u_i^2}$$

Basic idea: Normalize the units.

Tune from there.