Lecture 21 Algebraic Riccati Equation (ARE)

Our objective is to find conditions under which there exists a symmetric solution to the ARE

$$A^TP + PA + Q - (PB+N)^T R^{-1}(B^TP+N^T) = 0 \qquad (1.1)$$

where $A - BR^{-1}(B^TP+N^T)$ is Hurwitz.

Note that

$$A^TP + PA + Q - (PB+N)^T R^{-1}(B^TP+N^T) = 0$$

$$\iff A^TP + PA + Q - PBR^{-1}B^TP - PBR^{-1}N^T - NR^{-1}B^TP - NR^{-1}N^T = 0$$

$$\iff (A - BR^{-1}N^T)^TP + P(A - BR^{-1}N^T) + (Q - NR^{-1}N^T) - PBR^{-1}B^TP = 0$$

$$\iff \begin{bmatrix} P & -I \end{bmatrix} \begin{bmatrix} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & -(A - BR^{-1}N^T)^T \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = 0$$

$$\iff \begin{bmatrix} P & -I \end{bmatrix} H \begin{bmatrix} I \\ P \end{bmatrix} = 0$$

where

$$H \triangleq \begin{bmatrix} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & -(A - BR^{-1}N^T)^T \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$$

is called the <u>Hamiltonian Matrix</u> associated with ARE (1.1)

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It will turn out that the solution of the ARE is linked to properties (in particular eigenspaces) of the Hamiltonian Matrix.

We begin with the following claim:

<u>Claim 1</u> If λ is an eigenvalue of H, then $-\lambda$ is also an eigenvalue of H.

~~(ie since eigenvalues of real matrices ap~~

[Note that since eigenvalues of real matrices appear in complex conjugate pairs, the eigenvalues of H are symmetric across both the real and imaginary axes]

<u>proof</u>: Let $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and consider the similarity transformation JHJ^{-1}

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}}_{J^{-1}} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$JHJ^{-1} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & -(A - BR^{-1}N^T)^T \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -Q + NR^{-1}N^T & -(A - BR^{-1}N^T)^T \\ -(A - BR^{-1}N^T) & BR^{-1}B^T \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -(A - BR^{-1}N^T)^T & Q - NR^{-1}N^T \\ BR^{-1}B^T & A - BR^{-1}N^T \end{pmatrix} = - \begin{pmatrix} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & -(A - BR^{-1}N^T)^T \end{pmatrix}^T$$

$$= -H^T$$

Since eigenvalues are preserved under a similarity transformation, if λ is an eigenvalue of H, then it is an eigenvalue of $-H^T$, and since H and H^T have the same eigenvalues, λ is an eigenvalue of $-H$

(λ, x) - eigenpair of H

i.e. $\Rightarrow Hx = \lambda x \Rightarrow (-H)x = \lambda x$

$\Rightarrow H(-x) = (-\lambda)(-x)$

$\Rightarrow$ ~~$(-\lambda)$ is an eig~~

$(-\lambda, -x)$ - eigenpair of H

The next Lemma gives ~~us~~ conditions under which H does not have eigenvalues on the imaginary axis.

Lemma 21.1 Assume $Q - NR^{-1}N^T \geq 0$, and $R = R^T > 0$

If (1) (A, B) - stabilizable

(2) $(A - BR^{-1}N^T, Q - NR^{-1}N^T)$ - detectable

Then H has no eigenvalues on the imaginary axis.

proof: The proof will be by contradiction.

Assume that $j\omega$ is an eigenvalue of H with eigenvector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Then

$$(x_2^* \;\; x_1^*) H \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1^* \;\; x_2^*) H^T \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$= (x_2^* \;\; x_1^*)(Hx) + (Hx)^* \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$= \begin{pmatrix} x_2^* & x_1^* \end{pmatrix} j\omega \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \left(j\omega \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)^* \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$= j\omega\left(x_2^* x_1 + x_1^* x_2\right) - j\omega\left(x_1^* x_2 + x_2^* x_1\right)$$

$$= 0$$

Also $\Rightarrow$

$$\begin{pmatrix} x_2^* & x_1^* \end{pmatrix} H \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1^* & x_2^* \end{pmatrix} H^T \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$= \begin{pmatrix} x_2^* & x_1^* \end{pmatrix} \begin{pmatrix} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & -(A - BR^{-1}N^T)^T \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$+ \begin{pmatrix} x_1^* & x_2^* \end{pmatrix} \begin{pmatrix} (A - BR^{-1}N^T)^T & -Q + NR^{-1}N^T \\ -BR^{-1}B^T & -(A - BR^{-1}N^T) \end{pmatrix} \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$= \left(x_2^*(A - BR^{-1}N^T) + x_1^*(-Q + NR^{-1}N^T) \;\vdots\; -x_2^* BR^{-1}B^T - x_1^*(A - BR^{-1}N^T)^T \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$+ \left(x_1^*(A - BR^{-1}N^T)^T - x_2^* BR^{-1}B^T \;\vdots\; x_1^*(-Q + NR^{-1}N^T) - x_2^*(A - BR^{-1}N^T) \right) \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$= x_2^*(A - BR^{-1}N^T)x_1 + x_1^*(-Q + NR^{-1}N^T)x_1 - x_2^* BR^{-1}B^T x_2 - x_1^*(A - BR^{-1}N^T)^T x_2$$

$$+ x_1^*(A - BR^{-1}N^T)^T x_2 + x_1^*(-Q + NR^{-1}N^T)x_1 - x_2^* BR^{-1}B^T x_2 - x_2^*(A - BR^{-1}N^T)x_1$$

$$= -2x_1^*(Q - NR^{-1}N^T)x_1 - 2x_2^* BR^{-1}B^T x_2$$

$$= 0$$

Since $R > 0$ we must have

$$(Q - NR^{-1}N^T)x_1 = 0 \quad \text{and} \quad B^T x_2 = 0$$

Also $\left(j\omega, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$ - eigenpair of A implies that

$$\begin{pmatrix} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & -(A-BR^{-1}N^T)^T \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = j\omega\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} j\omega - (A-BR^{-1}N^T) & BR^{-1}B^T \\ Q - NR^{-1}N^T & j\omega + (A-BR^{-1}N^T)^T \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $$\left(j\omega - (A-BR^{-1}N^T)\right)x_1 + BR^{-1}B^Tx_2 = 0$$

$$\Rightarrow \left(j\omega - (A-BR^{-1}N^T)\right)x_1 = 0$$

$$\Rightarrow (A-BR^{-1}N^T)x_1 = j\omega x_1$$

~~So $(j\omega)$ not in left half plane for eigen with~~

So x_1 is an eigenvector of $A-BR^{-1}N^T$ that is orthogonal to $Q-NR^{-1}N^T$, and is associated with an eigenvalue not in open LHP

$\Rightarrow$ $(A-BR^{-1}N^T, Q-NR^{-1}N^T)$ is not ~~stabilizable~~ detectable $\#$

Also $$(Q-NR^{-1}N^T)x_1 + \left(j\omega + (A-BR^{-1}N^T)^T\right)x_2 = 0$$

$$\Rightarrow (A^T - NR^{-1}B^T)x_2 = j\omega x_2$$

$$\Rightarrow A^T x_2 = j\omega x_2$$

$\Rightarrow$ x_2 is an eigenvector of A^T that is in $\ker B$ ~~orthogonal to B~~ and is associated with ~~not~~ non-stable eigenvalue

$\Rightarrow (A,B)$ - not stabilizable $\#$

∴ We can conclude that if

(1) $Q - NR^{-1}N^T \geq 0$

(2) (A, B) - stabilizable

(3) $(A - BR^{-1}N^T, \; Q - NR^{-1}N^T)$ - detectable

then $H \in \mathbb{R}^{2n \times 2n}$ has exactly n eigenvalues in the open LHP and n eigenvalues in the open RHP

21.3 Stable Subspaces

Given a square matrix T, factor the characteristic polynomial as

$$\Delta(s) = \det(sI - T) = \Delta_-(s)\,\Delta_+(s)$$

where the roots of $\Delta_-(s)$ are in open LHP and $\Delta_+(s)$ are in closed RHP

The stable subspace is defined by

$$\mathcal{V}_- = \ker \Delta_-(T)$$

properties

P 21.1 $\dim \mathcal{V}_- = \deg \Delta_-(s)$

P 21.2 for every matrix V_- whose columns form a basis for $\mathcal{V}_-$, $\exists$ a stability matrix T_- s.t.

$$TV_- = V_- T_-$$

proof of P21.2

Decompose T as

$$T[W_- \; W_+] = [W_- \; W_+]\begin{pmatrix} J_- & 0 \\ 0 & J_+ \end{pmatrix}$$

where J_- is Jordan form associated with stable eigenvalues and W_- are associated eigenvalues, eigenvectors that span $\mathcal{V}_-$

$$\therefore \quad TW_- = W_- J_-$$

Now, for an invertible Z we have

$$TW_- Z^{-1} = W_- Z^{-1} Z J_- Z^{-1}$$

let $V_- = W_- Z^{-1}$ and $T_- = Z J_- Z^{-1}$, then

$$TV_- = V_- T_-$$

21.2 Domain of ~~Riccati~~ Riccati operator

Def: A Hamiltonian matrix $H \in \mathbb{R}^{2n \times 2n}$ is said to be in the domain of the Riccati operator if there exist a stabilizing matrix $H_- \in \mathbb{R}^{n \times n}$ and there exist a $P \in \mathbb{R}^{n \times n}$ s.t.

$$HM = MH_- \quad \text{where} \quad M = \begin{bmatrix} I \\ P \end{bmatrix}$$

Note: spans stable subspace

Thm 21.2

If $H \in Ric$, then

1) P satisfies the ARE

2) $H_- = A - BR^{-1}(B^T P + N^T)$

3) P is symmetric.

proof:

1)
$$HM = MH_-$$

$$\Leftrightarrow \quad H \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} H_-$$

$$\Rightarrow \quad [P \;\; -I] \, H \begin{bmatrix} I \\ P \end{bmatrix} = [P \;\; -I] \begin{bmatrix} I \\ P \end{bmatrix} H_- = (P - P) H_- = 0$$

Recall that the ARE can be written as

$$[P \;\; -I] \, H \begin{bmatrix} I \\ P \end{bmatrix} = 0$$

2) $$H \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} H_-$$

$$\Leftrightarrow \begin{bmatrix} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & -(A-BR^{-1}N^T)^T \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} H_-$$

The top line is

$$A - BR^{-1}(B^T P + N^T) = H_-$$

3)

$$HM = MH_-$$

$$\Rightarrow \underbrace{[-P^T \quad I] H \begin{bmatrix} I \\ P \end{bmatrix}}_{} = [-P^T \quad I]\begin{bmatrix} I \\ P \end{bmatrix} H_- = (P - P^T) H_-$$

in Homework you will
show this is symmetric
(plug + chug)

$\therefore$ $(P - P^T) H_-$ is symmetric

ie $$(P - P^T) H_- = H_-^T (P^T - P) = -H_-^T (P - P^T)$$

$$\Rightarrow (P - P^T) H_- + H_-^T (P - P^T) = 0$$

$$\Rightarrow e^{H_-^T t}(P - P^T) H_- e^{H_- t} + e^{H_-^T t} H_-^T (P - P^T) e^{H_- t} = 0 \quad \forall t$$

$$\Rightarrow \frac{d}{dt}\left\{ e^{H_-^T t}(P - P^T) e^{H_- t} \right\} = 0 \quad \forall t$$

$$\Rightarrow e^{H_-^T t}(P - P^T) e^{H_- t} \text{ is a constant for all } t$$

but since H_- is a stability matrix which converges to zero we must have that

$$e^{H_-^T t}(P - P^T)\, e^{H_- t} = 0 \qquad \text{for all } t$$

but since $e^{H_- t}$ and $e^{H_-^T t}$ are invertible,

$$P - P^T = 0 \Rightarrow P = P^T.$$

Putting it all together.

Thm 21.2 +

If
(i) ~~(A)~~ $R > 0$
(ii) ~~(B)~~ $Q - NR^{-1}N^T \geq 0$
(iii) ~~(C)~~ (A, B) - stabilizable
(iv) ~~(D)~~ $(A - BR^{-1}N^T,\ Q - NR^{-1}N^T)$ - ~~stabilizable~~ detectable

Then

1. H is in the domain of the Riccati operator.
2. P satisfies ARE
3. The solution of P is given by $V_1^{-1} V_2$ where $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ are the stable eigenvectors of H
4. $A - BR^{-1}(B^T P + N^T)$ is a stability matrix
   ie. $u = -Kx$ where $K = R^{-1}(B^T P + N^T)$ stabilizes $\dot{x} = Ax + Bu$
5. P is symmetric and positive definite.

proof:

1) Since H has n eigenvalues in the open LHP and n eigenvalues in the open RHP by claim 1 and lemma 21.1 we have

$$H(V_- \ V_+) = (V_- \ V_+)\begin{pmatrix} J_- & 0 \\ 0 & J_+ \end{pmatrix}$$

$$\Rightarrow \quad HV_- = V_- J_-$$

where $V_- = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ are the stable eigenvectors

If V_1^{-1} exists (come back to this later)

then

$$H V_- V_1^{-1} = V_- V_1^{-1} V_1 J_- V_1^{-1}$$

$$\Leftrightarrow \quad H\begin{pmatrix} I \\ V_2 V_1^{-1} \end{pmatrix} = \begin{pmatrix} I \\ V_2 V_1^{-1} \end{pmatrix} V_1 J_- V_1^{-1}$$

letting ~~where~~ $P = V_2 V_1^{-1}$ and $H_- = V_1 J_- V_1^{-1}$

gives $H\begin{pmatrix} I \\ P \end{pmatrix} = \begin{pmatrix} I \\ P \end{pmatrix} H_- \Rightarrow H \in \mathrm{Ric}$

2) Follows from Thm 21.2

3) We need to show that V_1^{-1} exists, or in other words a solution to ARE exists

(A,B) - stabilizable

$\Rightarrow \exists K$ s.t.

$$\dot{x} = (A-BK)x \text{ is exponentially stable}$$

$\Rightarrow x_k(t) \to 0$ exponentially

$\Rightarrow u_k(t) = -K x_k(t) \to 0$ exponentially

$$\Rightarrow \int_0^\infty x_k^T(t) Q x_k(t) + u_k^T R u_k + u_k^T N^T x_k(t) \, dt < \infty$$

but

$$x_0^T P x_0 = \min_u \int_0^\infty x^T Q x + u^T R u + u^T N^T x \, dt \le \int_0^\infty x_k^T Q x_k + u_k^T R u_k + u_k^T N^T x_k \, dt < \infty$$

for all x_0

$\therefore$ P exists $\Rightarrow$ V_1^{-1} exists.

4. Follows from Thm 21.2

5. P - symmetric follows from Thm 21.2

To show that P is symmetric;

The ARE can be written as

$$\cancel{A^T P + PA + Q}$$

$$(A - BR^{-1}N^T)^T P + P(A - BR^{-1}N^T) + Q - NR^{-1}N^T - PBR^{-1}B^T P = 0$$

$\Leftrightarrow$ $\cancel{A - B\pi}$

$$(A - BR^{-1}(B^T P + N^T))^T P + P(A - BR^{-1}(B^T P + N^T)) = -(Q - NR^{-1}N^T) - PBR^{-1}B^T P$$

$$H_-^T P + P H_- = -S$$

Recall from Thm 15.10 that ~~the~~ there exist a unique

pd P iff (H_-, S) - observable.

To show (H_-, S) observable, use eigenvector test by contradiction assume an eigenvector x of H_- that is in the $\ker(S)$ i.e. x satisfies

$$(A - BR^{-1}(B^T P + N^T))\, x = \lambda x \quad \text{and} \quad Sx = (Q - NR^{-1}N^T + PBR^{-1}B^T P)\, x = 0$$

Since both $Q - NR^{-1}N^T \geq 0$ and $PBR^{-1}B^T P \geq 0$ we must have $(Q - NR^{-1}N^T)\, x = 0$ and $B^T P x = 0$

$$\Rightarrow (A - BR^{-1}N^T)x - BR^{-1}B^T P x = (A - BR^{-1}N^T)x = \lambda x$$

$$\text{and} \quad (Q - NR^{-1}N^T)\, x = 0$$

~~[illegible]~~ $\Rightarrow (A - BR^{-1}N^T,\ Q - NR^{-1}N^T)$ - not stabilizable $\divideontimes$