Lecture 3

Definitions

$u \rightsquigarrow y$ means y is one of the outputs corresponding to input u, since different initial conditions produce different outputs

Causality

If $u \rightsquigarrow y$ then the system is causal in the sense that for every other input

$$\bar{u}(t) = u(t) \quad \forall\ 0 \le t \le T$$

the system exhibits at least one output $\bar{y}$ s.t.

$$\bar{y}(t) = y(t) \quad \forall\ 0 \le t \le T$$

– i.e. the output at time t does not depend on the future input

Time invariance

$$u(t) \longmapsto y(t) \implies u(t+T) \longmapsto y(t+T)$$

for every T.

– shifting the input, shifts the output.

Linearity

$$\begin{matrix} u_1 \rightsquigarrow y_1 \\ u_2 \rightsquigarrow y_2 \end{matrix} \implies \alpha u_1 + \beta u_2 \rightsquigarrow \alpha y_1 + \beta y_2$$

3.2 Characterizing all Outputs Corresponding to a given input

Theorem 3.1 Let $u \rightsquigarrow y_f$, then all outputs corresponding to u can be written as

$$y = y_f + y_h$$

where y_h is one of the output corresponding to the zero input

proof Since $u \rightsquigarrow y$ and $u \rightsquigarrow y_f$, then

$$0 = u - u \rightsquigarrow y - y_f = y_h .$$

Conversely if $u \rightsquigarrow y_f$ and $0 \rightsquigarrow y_h$

then $u + 0 \rightsquigarrow y_f + y_h$

$\therefore$ To construct all outputs corresponding to u

(1) Find one output corresponding to u &

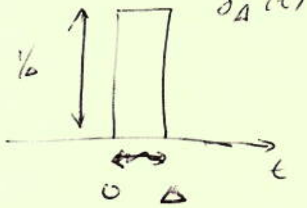
(e.g., zero initial conditions or particular solution)

(2) Find all outputs corresponding to the zero input

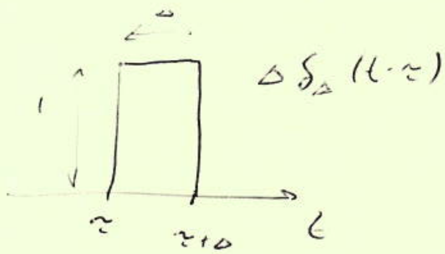
– homogeneous solution.

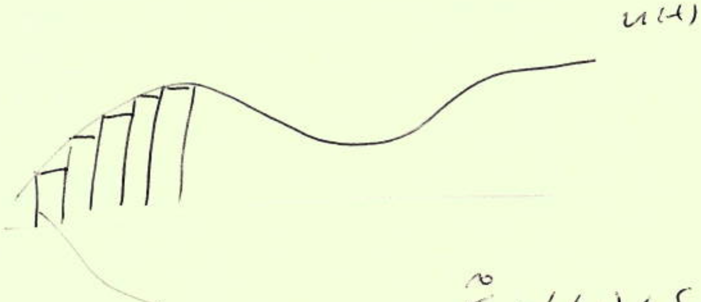
- Impulse response, transfer function, etc, solve (1)

Impulse Response



s.t. $\int_0^\infty \delta_\Delta(t)\,dt = 1$





$$u_\Delta(t) = \sum_{k=0}^{\infty} u(k\Delta)\,\Delta\,\delta_\Delta(t-k\Delta) \qquad t \geq 0$$

Let $\delta_\Delta(t-\tau) \rightsquigarrow g_\Delta(t,\tau)$

then $u_\Delta \rightsquigarrow y_\Delta(t) = \sum_{k=0}^{\infty} \Delta\, u(k\Delta)\, g_\Delta(t,k\Delta)$

Since $\lim_{\Delta\to 0} u_\Delta(t) = u(t)$ we have

$$u \rightsquigarrow y(t) = \lim_{\Delta\to 0} y_\Delta(t) = \int_0^\infty u(\tau)\, g(t,\tau)\, d\tau \qquad t \geq 0$$

where $g(t,\tau) = \lim_{\Delta\to 0} y_\Delta(t,\tau)$

$g(t,\tau)$ is the impulse response for a SISO system

For a mimo system we have

$$u \rightsquigarrow y(t) = \int_0^\infty G(t,\tau)\,u(\tau)\,d\tau \qquad \forall t \geq 0$$

where
$$G(t,\tau) = \begin{pmatrix} g_{11}(t,\tau) & \cdots & g_{1m}(t,\tau) \\ \vdots & & \\ g_{p1}(t,\tau) & \cdots & g_{pm}(t,\tau) \end{pmatrix}$$

and where $g_{ij}(t,\tau)$ is the impulse response from the j^{th} input to i^{th} output

<u>Properties</u>

1. $G(t,\tau) = 0 \quad \cancel{\forall \tau > t} \quad \forall t < \tau$ (impulse happens at τ)

proof: By causality.

2. For time invariant systems
$$G(t+T, \tau+T) = G(t,\tau) \qquad \forall t, \tau, T \geq 0$$

Note that if $\tau = 0$ and $t = t_2 - t_1$ and $T = t_1$

then
$$\begin{array}{ccc} G(t+T, \tau+T) & = & G(t,\tau) \\ \parallel & & \parallel \\ G(t_2 - t_1 + t_1, 0 + t_1) & & \\ \parallel & & \\ G(t_2, t_1) & = & G(t_2 - t_1, 0) \end{array}$$

3. For causal time-invariant systems
$$u \rightsquigarrow y = \int_0^\infty G(t,\tau)\,u(\tau)\,d\tau$$
$$= \int_0^t G(t,\tau)\,u(\tau)\,d\tau \qquad (\text{by } 1)$$

This is the well known convolution integral

The convolution integral

$$y(t) = \int_0^t G(t-\tau, 0)\, u(\tau)\, d\tau$$

gives the forced or particular solution,

3.4 <u>Laplace transform</u> (unilateral)

$$\mathcal{L}\{x(t)\} = \hat{X}(s) = \int_0^\infty e^{-st} x(t)\, dt \qquad s \in \mathbb{C}$$

- Laplace transform of derivative:

$$\mathcal{L}\{\dot{x}(t)\} = s\hat{X}(s) - x(0)$$

proof: $$\frac{d}{dt} e^{-st} x(t) = e^{-st}\dot{x}(t) - s e^{-st} x(t)$$

$$\Rightarrow \int_0^\infty \frac{d}{dt} e^{-st} x(t)\, dt = \int_0^\infty e^{-st}\dot{x}(t)\, dt - \int_0^\infty s e^{-st} x(t)\, dt$$

$$\int_0^\infty d\left(e^{-st} x(t)\right) = \lim_{t\to\infty} e^{-st} x(t) - e^{-s0} x(0) = 0 - x(0)$$

$$\mathcal{L}\{\dot{x}(t)\} - s\int_0^\infty e^{-st} x(t)\, dt = \mathcal{L}\{\dot{x}(t)\} - s\hat{X}(s)$$

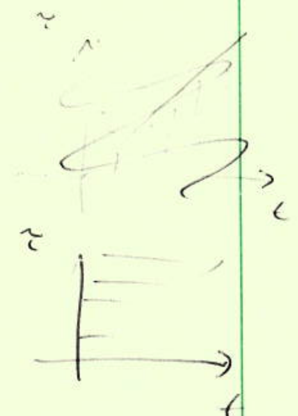
$$\therefore \mathcal{L}\{\dot{x}(t)\} = s\hat{X}(s) - x(0)$$

if the Laplace transform converges

Convolution

$$\mathcal{L}\{x * y\} = \mathcal{L}\left\{\int_0^t x(\tau)\, y(t-\tau)\, d\tau\right\} = \hat{x}(s)\,\hat{y}(s)$$

proof. $\mathcal{L}\left\{\int_0^t x(\tau)\, y(t-\tau)\, d\tau\right\}$



$$= \int_{t=0}^{\infty} e^{-st} \int_{\tau=0}^{t} x(\tau)\, y(t-\tau)\, d\tau\, dt$$

$$= \int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} e^{-st} e^{-s\tau} e^{s\tau} x(\tau)\, y(t-\tau)\, dt\, d\tau$$

$$= \int_{\tau=0}^{\infty} e^{-s\tau} x(\tau) \int_{t=\tau}^{\infty} e^{-s(t-\tau)} y(t-\tau)\, dt\, d\tau$$

Let $\sigma = t - \tau$

$$= \int_{\tau=0}^{\infty} e^{-s\tau} x(\tau)\, d\tau \int_{\sigma=0}^{\infty} e^{-s\sigma} y(\sigma)\, d\sigma$$

$$= \hat{x}(s)\,\hat{y}(s)$$

Transfer function