Lecture 4

~~Lature~~

Transfer function for state space systems

Continuous-time

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

taking the Laplace transform gives

$$s\hat{X}(s) - x(0) = A\hat{x}(s) + B\hat{u}(s)$$
$$\hat{y}(s) = C\hat{x}(s) + D\hat{u}(s)$$

$$\Rightarrow (sI - A)\hat{x}(s) = x(0) + B\hat{u}(s)$$
$$\Rightarrow \hat{x}(s) = (sI-A)^{-1}x(0) + (sI-A)^{-1}B\hat{u}(s)$$
$$\Rightarrow \hat{y}(s) = C(sI-A)^{-1}x(0) + \left[C(sI-A)^{-1}B + D\right]\hat{u}(s)$$

Let $\hat{G}(s) = C(sI-A)^{-1}B + D$ ← transfer function

$\hat{\Psi}(s) = C(sI-A)^{-1}$

then

$$\hat{y}(s) = \hat{\Psi}(s)\,x(0) + \hat{G}(s)\,\hat{u}(s)$$

Inverse Laplace transform to get

$$y(t) = \Psi(t)\,x_0 + \int_0^t G(t-\tau)u(\tau)\,d\tau$$

where $G(t) = \mathcal{L}^{-1}\{\hat{G}(s)\}$ is the impulse response

Discrete-time

$$x^+ = Ax + Bu$$
$$y = Cx + Du$$

taking z-transform

$$z\hat{X}(z) - x(0) = A\hat{X}(z) + Bu(z)$$
$$\hat{y}(z) = Cx(\hat{z}) + Du(z)$$

$\Rightarrow$

$$\hat{y}(z) = C(zI-A)^{-1}x(0) + \left[C(zI-A)^{-1}B + D\right]u(z)$$

the transfer function is

$$G(z) = C(zI-A)^{-1}B + D$$

and the impulse response is

$$G(t) = Z^{-1}\{C(zI-A)^{-1}B + D\}$$

Realization theory

$$\left.\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}\right\}$$ is said to be a realization of $\hat{G}(s)$ if

$$\hat{G}(s) = C(sI-A)^{-1}B + D$$

Def: ~~Zero state~~ Two state space systems are said to be "zero-state equivalent" if they realize the same transfer function.

(same zero-state response, & but could have different initial condition response!)

Example

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = (4 \quad 5)\, x$$

$$\hat{G}(s) = (4 \quad 5)\left(s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}\right)^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= (4 \quad 5)\begin{pmatrix} s & -1 \\ 3 & s+2 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{(4 \quad 5)\begin{pmatrix} s+2 & 1 \\ -3 & s \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{s^2+2s+3}$$

$$= \frac{(4 \quad 5)\begin{pmatrix} 1 \\ s \end{pmatrix}}{s^2+2s+3}$$

$$= \frac{5s+4}{s^2+2s+3}$$

In taking the inverse we used the formula

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} \leftarrow \text{transpose of the cofactors of } A$$

(adjugate of **A** - not adjoint like it says in book)

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$\operatorname{adj}(A) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\det(A) = ad - bc$$

$$\therefore \; A^{-1} = \frac{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}{ad - bc}$$

<u>Example continued</u>

$$\dot{x} = \begin{pmatrix} -3 & 1 \\ -6 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u$$

$$y = \begin{pmatrix} 4 & 1/2 \end{pmatrix} x$$

$$\hat{G}(s) = \begin{pmatrix} 4 & 1/2 \end{pmatrix} \begin{pmatrix} s+3 & -1 \\ 6 & s-1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \frac{\begin{pmatrix} 4 & 1/2 \end{pmatrix} \begin{pmatrix} s-1 & 1 \\ -6 & s+3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}}{s^2 + 2s + 3}$$

$$= \frac{\begin{pmatrix} 4 & 1/2 \end{pmatrix} \begin{pmatrix} s+1 \\ 2s \end{pmatrix}}{s^2 + 2s + 3}$$

$$= \frac{5s + 4}{s^2 + 2s + 3}$$

The two systems are zero-state equivalent.

Note that since

$$\hat{G}(s) = C(sI-A)^{-1}B + D$$

$$= \frac{C\,\mathrm{adj}(sI-A)\,B}{\det(sI-A)} + D$$

$$= \frac{C\,\mathrm{adj}(sI-A)\,B + D\det(sI-A)}{\det(sI-A)}$$

Note that $\det(sI-A) = 0$ is an n^{th} order polynomial

Also since we cancel one row and column in computed each element of $\mathrm{adj}(sI-A)$, ~~then~~ each element is at most an $(n-1)^{th}$ order polynomial

$\therefore$ $C\,\mathrm{adj}(sI-A)\,B$ is a matrix of polynomials whose order is ~~less than~~ at most $n-1$. and

$C\,\mathrm{adj}(sI-A)B + D\det(sI-A)$ is a matrix of polynomials whose order is at most n

If $D=0$ then we say that $\hat{G}(s)$ is a

"strictly proper rational function"

rational because ratio of polynomials

If $D \neq 0$ then we say that $\hat{G}(s)$ is a

"proper rational function"

Theorem 4.3 A transfer function $\hat{G}(s)$ can be realized by an LTI state space system iff $\hat{G}(s)$ is a proper rational function.

Given a state space ~~system~~ realization we have seen how to obtain the transfer matrix.

Thm 4.3 says we can go the other way.

Is there a constructive method to do so?

Step 1 Given $\underset{m \times k}{\hat{G}(s)}$

Decompose

$$\hat{G}(s) = \underbrace{\hat{G}_{sp}(s)}_{\text{strictly proper}} + D$$

where $D = \lim_{s \to \infty} \hat{G}(s)$ and

$$\hat{G}_{sp}(s) = C(sI - A)^{-1}B$$

Focus on finding A, B, C

Step 2

Find monic least common denominator for $\hat{G}_{sp}(s)$

↑ leading 1

$$d(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s^0 + \alpha_n$$

Example

$$\hat{G}(s) = \begin{pmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{pmatrix}$$

$$D = \lim_{s \to \infty} \begin{pmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{pmatrix}$$

$$= \lim_{s \to \infty} \begin{pmatrix} \frac{4 - 10/s}{2 + 1/s} & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

Since

$$2 + G_{11} = \frac{4s-10}{2s+1}$$

$$\Rightarrow G_{11} = \frac{4s-10}{2s+1} - \frac{2(2s+1)}{2s+1}$$

$$= \frac{4s - 10 - 4s - 2}{2s+1}$$

$$= \frac{-12}{2s+1}$$

$$\Rightarrow \hat{G}_{sp}(s) = \begin{pmatrix} \frac{-12}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{pmatrix}$$

$$d'(s) = (2s+1)(s+2)^2$$

$$= (2s+1)(s^2 + 4s + 4)$$

$$= 2s^3 + 9s^2 + 12s + 4$$

not monic so divide by 2

$\cancel{d(s) = s^3 + 9s^2 + 12s + 4}$

$$d(s) = s^3 + 9/2\, s^2 + 6s + 2$$

Step 3

Expand $\hat{G}_{sp}(s)$ as

$$\hat{G}_{sp}(s) = \frac{1}{d(s)}\left[N_1 s^{n-1} + N_2 s^{n-2} + \dots + N_n\right]$$

Example

$$G_{sp}(s) = \frac{\begin{pmatrix} \frac{-12(s+2)^2}{2} & \frac{3}{3}(s+\frac{1}{2})(s+2) \\ \frac{s+2}{2} & (s+1)(s+\frac{1}{2}) \end{pmatrix}}{s^3 + \frac{9}{2}s^2 + \frac{12}{6}s + \frac{4}{2}}$$

$$= \frac{\begin{pmatrix} -6(s^2+4s+4) & 3(s^2+\frac{3}{2}s+1) \\ \frac{s+2}{2} & s^2+\frac{3}{2}s+\frac{1}{2} \end{pmatrix}}{s^3 + \frac{9}{2}s^2 + \frac{12}{2}s + \frac{4}{2}}$$

$$= \frac{\begin{pmatrix} -6 & 3 \\ 0 & 1 \end{pmatrix} s^2 + \begin{pmatrix} -24 & 9/2 \\ 1/2 & 3/2 \end{pmatrix} s + \begin{pmatrix} -24 & 3 \\ 1 & 1/2 \end{pmatrix}}{s^3 + \frac{9}{2}s^2 + \frac{12}{2}s + \frac{4}{2}}$$

$$N_1 = \begin{pmatrix} -6 & 3 \\ 0 & 1 \end{pmatrix} \quad N_2 = \begin{pmatrix} -24 & 9/2 \\ 1/2 & 3/2 \end{pmatrix}$$

$$N_3 = \begin{pmatrix} -24 & 3 \\ 1 & 1/2 \end{pmatrix}$$

Step 4

Create an analog computer implementation of $\hat{G}_{sp}(s)$

$$Y_{sp}(s) = \left[N_1 s^{n-1} + \dots + N_n\right]\frac{U(s)}{d(s)}$$

Assume $Y_{sp}(s)$ is of the form

$$\left[N_1 s^{n-1} + \dots N_n\right] Z(s)$$

then

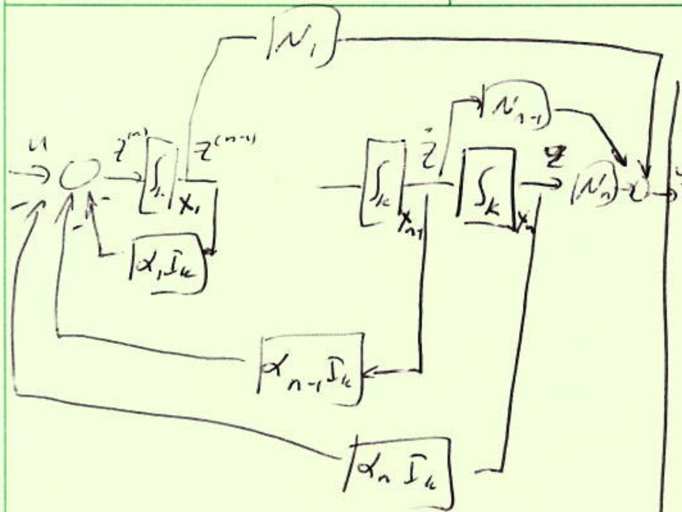
$$\left[N_1 s^{n-1} + \dots + N_n\right] \underbrace{Z(s)}_{r\times 1} = \left[N_1 s^{n-1} + \dots + N_n\right]\frac{U(s)}{d(s)}$$

$$\Rightarrow \quad Z(s) = \frac{U(s)}{d(s)}$$

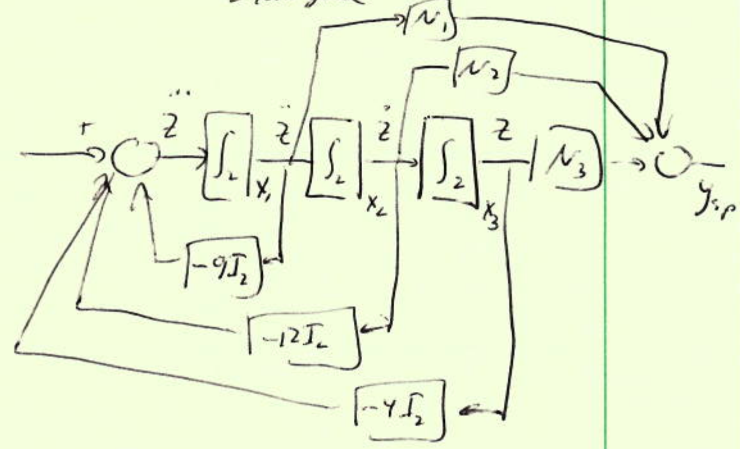
or

$$\left(s^n + \alpha_1 s^{n-1} + \dots + \alpha_n\right) Z(s) = U(s)$$

$$\Rightarrow \quad Z^{(n)} + \alpha_1 Z^{(n-1)} + \dots + \alpha_n Z = U$$



Example



label the output of the integrators as states + write the state space equations

$$\begin{pmatrix}\dot X_1\\ \dot X_2\\ \vdots\\ \dot X_{n-1}\\ \dot X_n\end{pmatrix} = \underbrace{\begin{pmatrix} -\alpha_1 I_k & -\alpha_2 I_k & \cdots & -\alpha_n I_k \\ I_k & & & \\ & \ddots & & \\ & & I_k & 0\end{pmatrix}}_{A}\begin{pmatrix}X_1\\ X_2\\ \vdots \\ X_n\end{pmatrix} + \underbrace{\begin{pmatrix} I_k\\ 0\\ \vdots \\ 0\end{pmatrix}}_{B} u$$

$$y = \underbrace{\begin{pmatrix} N_1 & N_2 & \cdots & N_n\end{pmatrix}}_{C}\begin{pmatrix}X_1\\ \vdots\\ X_n\end{pmatrix}$$

$$\begin{pmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\end{pmatrix} = \begin{pmatrix} -\frac{9I_2}{2} & -\frac{12}{2}I_2 & -\frac{4}{2}I_2 \\ I_2 & 0_2 & 0_2 \\ 0_2 & I_2 & 0_2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} + \begin{pmatrix} I_2\\ 0\\ 0\end{pmatrix} u$$

$$y = \begin{pmatrix} N_1 & N_2 & N_3\end{pmatrix} X$$

$\therefore$

$$A = \begin{pmatrix} -9/2 & 0 & -6 & 0 & -2 & 0 \\ 0 & -9/2 & 0 & -6 & 0 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} -6 & 3 & -24 & 9/2 & -24 & 3 \\ 0 & 1 & 1/2 & 3/2 & 1 & 1/2 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$