Lecture 5   L. Time - Varying Systems

Thm 5.1 The unique solution to the ~~initial condition~~ zero input ~~system~~ time-varying system

$$\dot{x} = A(t)\,x \qquad x(t_0) = x_0 \qquad t \geq 0$$

is given by

$$x(t) = \Phi(t, t_0)\, x_0$$

where

$$\Phi(t,\tau) = I + \int_{\tau}^{t} A(s_1)\,ds_1$$

$$+ \int_{\tau}^{t} A(s_1) \int_{\tau}^{s_1} A(s_2)\,ds_2\,ds_1$$

$$+ \int_{\tau}^{t} A(s_1) \int_{\tau}^{s_1} A(s_2) \int_{\tau}^{s_3} A(s_3)\,ds_3\,ds_2\,ds_1$$

$$+ \cdots$$

↑ State transition matrix.

Peano - Baker Series

The proof follows from the next lemma

Lemma P5.1 For every $t \geq 0$ $\Phi(t, t_0)$ is the unique solution to

$$\frac{d}{dt}\Phi(t,t_0) = A(t)\,\Phi(t,t_0) \quad , \quad \Phi(t_0, t_0) = I$$

Proof:

The proof of P5.1 depends on Leibnitz rule

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,x)\,dx = f(t,b(t))\frac{db}{dt} - f(t,a(t))\frac{da}{dt} + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(t,x)\,dx$$

proof of P5.1

$$\frac{d}{dt}\Phi(t,t_0) = \frac{d}{dt}\Big[I + \int_{t_0}^{t} A(s_1)\,ds_1 + \int_{t_0}^{t} A(s_1)\int_{t_0}^{s_1} A(s_2)\,d\sigma_2\,d\sigma_1 + \int_{t_0}^{t} A(s_1)\int_{t_0}^{s_1} A(s_2)\int_{t_0}^{s_2} A(s_3)\,d\sigma_3\,d\sigma_2\,ds_1 + \cdots \Big]$$

$$= 0 + A(t) + A(t)\int_{t_0}^{t} A(s_2)\,d\sigma_2 + A(t)\int_{t_0}^{t} A(s_2)\int_{t_0}^{s_2} A(s_3)\,d\sigma_3\,d\sigma_2 + \cdots \Big]$$

$$= A(t)\Big[I + \int_{t_0}^{t} A(s_1)\,d\sigma_1 + \int_{t_0}^{t} A(s_1)\int_{t_0}^{s_1} A(s_2)\,d\sigma_2\,ds_1 + \cdots \Big]$$

$$= A(t)\,\Phi(t,t_0)$$

Also note that

$$\Phi(t_0,t_0) = I + \int_{t_0}^{t_0} A(s_1)\,d\sigma_1 + \int_{t_0}^{t_0} A(s_1)\int_{t_0}^{s_1} A(s_2)\,d\sigma_2\,d\sigma_1 + \cdots$$

$$= I$$

Proof of thm 5.1

Let $x(t) = \Phi(t,t_0)\,x_0$

then $x(t_0) = \Phi(t_0,t_0)\,x_0 = x_0$

and $$\dot{x} = \frac{d}{dt}\Phi(t,t_0)\,x_0$$

$$= A(t)\,\Phi(t,t_0)\,x_0$$

$$= A(t)\,x(t)$$

Property 5.3 (Semi-group property)

for every $t, s, \tau \geq 0$

$$\Phi(t,s)\,\Phi(s,\tau) = \Phi(t,\tau)$$

proof:

Let x_0 be arbitrary, then the solution to

$$\dot{x} = A(t)\,x \qquad x(\overset{\tau}{\cancel{t_0}}) = x_0 \qquad \text{at times } s < t$$

s, t and t_0

$$x_1 \overset{\circ}{=} \Phi(s,\tau)\,x_0 \qquad x_2 \overset{\circ}{=} \Phi(t,\tau)\,x_0$$

however x_2 is also the solution to

$$x_2 = \Phi(t,s)\,x_1$$

so $$x_2 = \Phi(t,s)\,\Phi(s,\tau)\,x_0 = \Phi(t,\tau)\,x_0$$

Since x_0 is arbitrary we have

$$\Phi(t,s)\,\Phi(s,\tau) = \Phi(t,\tau)$$

Property 5.4 for every $t, \tau \geq 0$ $\Phi(t,\tau)$ is nonsingular and

$$\Phi^{-1}(t,\tau) = \Phi(\tau,t)$$

From property 5.3

$$\Phi(t,\tau)\,\Phi(\tau,t) = \Phi(\tau,t)\,\Phi(t,\tau) = I$$

$$\therefore \quad \Phi^{-1}(t,\tau) = \Phi(\tau,t)$$

~~Thm~~

Thm 5.2

The unique solution of

$$\dot{x} = A(t)x + B(t)u \quad , y(t) = C(t)x + D(t)u \ , x(t_0) = x_0$$

is

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)\cancel{d\tau}\,u(\tau)\,d\tau \qquad (4.1)$$

$$y(t) = C(t)\Phi(t,t_0)x_0 + \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau \quad D(t)u(t) \qquad 4.2)$$

proof Differentiating 4.1 gives

$$\dot{x} = A(t)\Phi(t,t_0)x_0 + \Phi(t,t)B(t)u(t) + \int_{t_0}^{t} A(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

$$= A(t)\left[\Phi(t,t_0)x_0 + \int_{t_0}^{t}\Phi(t,\tau)B(\tau)u(\tau)\,d\tau\right] + B(t)u(t)$$

$$= A(t)x(t) + B(t)u(t)$$

~~Also~~ $\Rightarrow y(t)$ is given by (4.2)

To check initial conditions, not that

$$x(t_0) = \Phi(t_0,t_0)x_0 + \int_{t_0}^{t_0}\{\text{stuff}\}\,d\tau$$

$$= x_0$$

Discrete - Time

First consider

$$x(t+1) = A(t)\,x(t) \quad , \quad x(t_0) = x_0$$

so

$$x(t_0+1) = A(t_0)\,x(t_0) = A(t_0)\,x_0$$

$$x(t_0+2) = A(t_0+1)\,x(t_0+1)$$

$$= A(t_0+1)\,A(t_0)\,x_0$$

$$x(t_0+3) = A(t_0+2)\,x(t_0+2)$$

$$= A(t_0+2)\,A(t_0+1)\,A(t_0)\,x_0$$

$$\vdots$$

$$x(t) = A(t-1)\,A(t-2)\cdots A(t_0+2)\,A(t_0+1)\,A(t_0)\,x_0$$

$\therefore$ Define

$$\Phi(t,t_0) = \begin{cases} I\,, & t = t_0 \\ A(t-1)\,A(t-2)\cdots A(t_0+1)\,A(t_0)\,, & t > t_0 \end{cases}$$

then

$$x(t) = \Phi(t,t_0)\,x_0 \qquad t \geq t_0$$

Note $t \geq t_0$ for discrete time → can't reverse time in the discrete time, but you can in ~~the~~ continuous time!

Thm 5.3 The unique solution of

$x(t+1) = A(t)x(t) + B(t)u(t)$, $y(t) = C(t)x(t) + D(t)u(t)$, $x(t_0) = x_0$

is

$$x(t) = \Phi(t,t_0)x_0 + \sum_{\tau=t_0}^{t-1} \Phi(t,\tau+1)B(\tau)u(\tau) \qquad t \geq t_0 \qquad (6.1)$$

$$y = C(t)\Phi(t,t_0)x_0 + \sum_{\tau=t_0}^{t-1} \Phi(t,\tau+1)B(\tau)u(\tau) + D(t)u(t) \qquad t \geq t_0$$

proof: (6.1) By induction

$$x(t_0) = \Phi(t_0,t_0)x_0 + \sum_{\tau=t_0}^{t_0-1} [blah]$$

$$= Ix_0 = x_0$$

Assume true for t ie ~~$x(t-1) = A(t)x$~~

~~$x(t) = A(t-1)x(t-1) +$~~

$$x(t+1) = \Phi(t+1,t_0)x_0 + \sum_{\tau=t_0}^{t} \Phi(t+1,\tau+1)B(\tau)u(\tau)$$

$$= \Phi(t+1,t)\Phi(t,t_0)x_0 + \sum_{\tau=t_0}^{t-1} \Phi(t+1,t)\Phi(t,\tau+1)B(\tau)u(\tau)$$

$$+ \Phi(t+1,t+1)B(t)u(t)$$

$$= \Phi(t+1,t)\left[\Phi(t,t_0)x_0 + \sum_{\tau=t_0}^{t-1} \Phi(t,\tau+1)B(\tau)u(\tau)\right]$$

$$+ \Phi(t+1,t+1)B(t)u(t)$$

$$= \Phi(t+1,t)x(t) + B(t)u(t)$$

But $\Phi(t+1,t) = A(t)$

P5.5 for every $t \geq t_0$, $\Phi(t,t_0)$ is the unique solution to

$$\phi(t+1, t_0) = A(t)\Phi(t,t_0), \quad \Phi(t_0, t_0) = I \quad t \geq t_0$$

proof: - obvious from definition of $\Phi(t,t_0)$

P5.7 for every $t \geq s \geq \tau \geq 0$

$$\Phi(t,s)\Phi(s,\tau) = \Phi(t,\tau)$$

Thm 5.3 The unique soln of

$$x(t+1) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t) \quad x(t_0) = x_0$$

is

$$x(t) = \Phi(t,t_0)x_0 + \sum_{\tau=t_0}^{t-1} \Phi(t,\tau+1)B(\tau)u(\tau) \quad t \geq t_0 \quad (6.1)$$

$$y(t) = C(t)\Phi(t,t_0)x_0 + \sum_{\tau=t_0}^{t-1} C(t)\Phi(t,\tau+1)B(\tau)u(\tau) + D(t)u(t) \quad 6.2$$

proof: From 6.1 we have

$$x(t+1) = \Phi(t+1,t_0)x_0 + \sum_{\tau=t_0}^{t} \Phi(t+1,\tau+1)B(\tau)u(\tau)$$

$$= \Phi(t+1,t)\Phi(t,t_0)x_0 + \sum_{\tau=t_0}^{t-1} \Phi(t+1,\tau+1)B(\tau)u(\tau)$$

$$+ \Phi(t+1,t+1)B(t)u(t)$$

$$= \Phi(t+1,t)\Phi(t,t_0)x_0 + \sum_{\tau=t_0}^{t-1} \Phi(t+1,t)\Phi(t,\tau+1)B(\tau)u(\tau) + B(t)u(t)$$

$$= \Phi(t+1,t)\left[\Phi(t,t_0)x_0 + \sum_{\tau=t_0}^{t-1} \Phi(t,\tau+1)B(\tau)u(\tau)\right] + B(t)u(t)$$

$$= \Phi(t+1,t) A x(t) + B(t)u(t)$$

but

$$\Phi(t+1,t) = A(t)$$