Lecture 6

Solutions to LTI state space systems

Recall derivation of 1st order scalar diff eq.

$$\dot{y} = ay + bu$$

Move y's + u's to same side of equation:

$$\dot{y} - ay = bu$$

Introduce integrating factor

$$\underbrace{e^{-at}(\dot{y} - ay)} = e^{-at} bu$$

$$\frac{d}{dt} e^{-at} y = e^{-at} bu \quad \leftarrow \text{because } \frac{d}{dt} e^{-at} = e^{-at}(-a)$$

Integrate both sides from t_0 to t

$$\int_{t_0}^{t} d(e^{-at} y) = \int_{\tau = t_0}^{t} e^{-a\tau} b u(\tau) \, d\tau$$

$$\Rightarrow \quad e^{-at} y(t) - e^{-at_0} y(t_0) = \int_{\tau = t_0}^{t} e^{-a\tau} b u(\tau) \, d\tau$$

Multiply both sides by e^{at} to get

$$y(t) = e^{a(t-t_0)} y(t_0) = \int_{\tau = t_0}^{t} e^{a(t-\tau)} b u(\tau) \, d\tau \quad \leftarrow \text{because } e^{a} e^{b} = e^{a+b}$$

Let's try to do the same thing for the state space eq

$$\dot{x} = Ax + Bu \quad , \quad y = Cx + Du \quad , \quad x(t_0) = x_0$$

Move x's + u's to same side of equation:

$$\dot{x} - Ax = Bu$$

We need an integrating factor with the properties that

(1) $\frac{d}{dt} e^{-At} = e^{-At}(-A)$

(2) $e^{At} e^{A\tau} = e^{A(t+\tau)}$

Recall that the exponential with scalar argument is defined as

$$e^{at} = 1 + at + \frac{1}{2!}a^2t^2 + \frac{1}{3!}a^3t^3 + \cdots$$

$$\Rightarrow \frac{d}{dt}e^{at} = \frac{d}{dt}\left(1 + at + \frac{1}{2!}a^2t^2 + \frac{1}{3!}a^3t^3 + \cdots\right)$$

$$= a + a^2t + \frac{1}{2!}a^3t^2 + \cdots)$$

$$= a(1 + at + \frac{1}{2!}a^2t^2 + \cdots) = (1 + at + \frac{1}{2!}a^2t^2 + \cdots)a$$

$$= ae^{at} \qquad = e^{at}a$$

and by direct multiplication we can show that

$$e^{at}e^{a\tau} = (1 + at + \frac{1}{2!}a^2t^2 + \cdots)(1 + a\tau + \frac{1}{2!}a\tau^2 + \cdots)$$

$$= 1 + a(t+\tau) + \frac{1}{2!}a^2(t+\tau)^2 + \cdots)$$

$$= e^{a(t+\tau)}$$

Motivated by these properties, ~~let~~ define

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

where $A^2 = AA$

$A^3 = AAA$

Note that

$$\frac{d}{dt} e^{At} = \frac{d}{dt}\left(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right)$$

$$= A + A^2t + \frac{1}{2!}A^3t^2 + \cdots$$

$$= A\left(I + At + \frac{1}{2!}A^2t^2 + \cdots\right) = \left(I + At + \frac{1}{2!}A^2t^2 + \cdots\right)A$$

$$= \underbrace{A e^{At} = e^{At} A}_{\text{A commutes with its exponential!}}$$

Also by direct comparison to the scalar case we have

$$e^{At} e^{A\tau} = e^{A(t+\tau)}$$

Back to solving the ODE:

$$\underbrace{e^{-At}\left(\dot{x} - Ax\right)} = e^{-At}Bu$$

$$\frac{d}{dt}\left(e^{-At}x\right) = e^{-At}Bu(t)$$

$$\Rightarrow \int_{t_0}^{t} d\left(e^{-At}x\right) = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)\,d\tau$$

$$\Rightarrow e^{-At}x(t) - e^{-At_0}x(t_0) = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)\,d\tau$$

$$\Rightarrow x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$\Rightarrow y(t) = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Cu(t)$$

In the previous chapter we saw that the solution to the LTV system $\dot{x} = A(t)x$ ~~$+ B(t)u$~~, $x(t_0)$ was given by

$$x(t) = \Phi(t,t_0)\,x(t_0) \quad \text{where}$$

$$\Phi(t,t_0) = I + \int_{t_0}^{t} A(s)\,ds + \int_{s_1=t_0}^{t} A(s_1)\int_{s_2=t_0}^{s_1} A(s_2)\,ds_2\,ds_1 + \int_{s_1=t_0}^{t} A(s_1)\int_{s_2=t_0}^{s_1} A(s_2)\int_{s_3=t_0}^{s_2} A(s_3)\,ds_3\,ds_2\,ds_1 + \dots$$

Let $t_0 = 0$ and let $A(t) \equiv A$, then

$$\int_{t_0}^{t} A(s)\,ds = A\int_{0}^{t} ds = At$$

$$\int_{s_1=t_0}^{t} A(s_1)\int_{s_2=t_0}^{s_1} A(s_2)\,ds_2\,ds_1 = A^2\int_{s_1=0}^{t}\int_{s_2=0}^{s_1} ds_2\,ds_1 = \frac{1}{2!}t^2$$

$$\Rightarrow \quad \Phi(t,0) = e^{At} \qquad \text{or} \qquad \Phi(t,\tau) = e^{A(t-\tau)}$$

Note that $e^{\begin{pmatrix}1&2\\3&4\end{pmatrix}t} \neq \begin{pmatrix} e^t & e^{2t}\\ e^{3t} & e^{4t}\end{pmatrix}$!

proof

$$e^{\begin{pmatrix}1&2\\3&4\end{pmatrix}t} = I + \begin{pmatrix}1&2\\3&4\end{pmatrix}t + \frac{1}{2!}\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}t^2 + \frac{1}{3!}\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}t^3 + \cdots$$

$$= \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&2\\3&4\end{pmatrix}t + \frac{1}{2!}\begin{pmatrix}7&10\\15&22\end{pmatrix}t^2 + \frac{1}{3!}\begin{pmatrix}37&54\\81&118\end{pmatrix}t^3 + \cdots$$

$$= \begin{pmatrix} 1+t+\frac{1}{2!}7t^2+\frac{1}{3!}37t^3+\cdots & 2t+\frac{1}{2!}10t^2+\frac{1}{3!}54t^3+\cdots \\ 3t+\frac{1}{2!}15t^2+\frac{1}{3!}81t^3+\cdots & 1+4t+\frac{1}{2!}22t^2+\cdots \end{pmatrix}$$

$$\begin{pmatrix}e^t & e^{2t}\\ e^{3t} & e^{4t}\end{pmatrix} = \begin{pmatrix} 1+t+\frac{1}{2!}t^2+\frac{1}{3!}t^3+\cdots & 1+2t+\frac{1}{2!}4t^2+\frac{1}{3!}8t^3+\cdots \\ 1+3t+\frac{1}{2!}9t^2+\frac{1}{3!}27t^3+\cdots & 1+4t+\frac{1}{2!}16t^2+\frac{1}{3!}64t^3+\cdots \end{pmatrix}$$

We will discuss three methods for computing e^{At}.

Method 1 (Laplace transform)

Solution of $\dot{X} = AX$, $X(0) = I$ is

$$X(t) = e^{At}$$

taking the Laplace transform gives

$$s\hat{X}(s) - I = A\hat{X}(s)$$

$$\Rightarrow \quad (sI - A)\hat{X}(s) = I$$

$$\Rightarrow \quad \hat{X}(s) = (sI - A)^{-1}$$

$$\Rightarrow \quad X(t) = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

$$\therefore \quad \boxed{e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}}$$

Example

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\} = \mathcal{L}^{-1}\left\{ \begin{pmatrix} s & -1 \\ 2 & s+3 \end{pmatrix}^{-1} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{\begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix}}{s^2 + 3s + 2} \right\} = \mathcal{L}^{-1}\left\{ \begin{pmatrix} \frac{s+3}{(s+2)(s+1)} & \frac{1}{(s+2)(s+1)} \\ \frac{-2}{(s+2)(s+1)} & \frac{s}{(s+2)(s+1)} \end{pmatrix} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \begin{pmatrix} \frac{2}{s+1} + \frac{-1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{pmatrix} \right\}$$

$$= \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}$$

Method 2 (Eigenvalue / Eigenvector Method)

Theory. Suppose that the geometric multiplicity = algebraic multiplicity

then $$AP = P\Lambda \Rightarrow A = P\Lambda P^{-1}$$

where Λ is diagonal (and P + Λ are complex)

Note that
$$A^2 = AA = P\Lambda P^{-1} P\Lambda P^{-1} = P\Lambda^2 P^{-1}$$
$$A^3 = AA^2 = P\Lambda P^{-1} P\Lambda^2 P^{-1} = P\Lambda^3 P^{-1}$$
$$\Rightarrow A^k = P\Lambda^k P^{-1}$$

$$\Rightarrow e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$
$$= PP^{-1} + P\Lambda P^{-1}t + \frac{1}{2!}P\Lambda^2P^{-1}t^2 + \frac{1}{3!}P\Lambda^3P^{-1}t^3 + \cdots$$
$$= P\underbrace{\left(I + \Lambda t + \frac{1}{2!}\Lambda^2t^2 + \frac{1}{3!}\Lambda^3t^3 + \cdots\right)}P^{-1}$$
$$= Pe^{\Lambda t}P^{-1}$$

and since Λ is a diagonal matrix we have that

$$e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} \text{ where } \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Example:

Example: $\quad A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$

$$\det(sI - A) = \det\begin{pmatrix} s & -1 \\ 2 & s+3 \end{pmatrix} = s^2 + 3s + 2 = (s+2)(s+1)$$

$\therefore$ eigenvalues are $\Lambda = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$

eigenvector associated w/ $\lambda_1 = -2$:

$$\begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2x_1 = -x_2 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

eigenvector associated w/ $\lambda_1 = -1$

$$\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -x_1 - x_2 = 0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore P = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \quad \text{and} \quad P^{-1} = \frac{\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}}{-1+2} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$\therefore e^{At} = P e^{\Lambda t} P^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-2t} & e^{-t} \\ -2e^{-2t} & -e^{-t} \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}$$

Method 3 (Cayley - Hamilton)

Theory:

~~Thm 6.1 (Cayley-Hamilton)~~

Def: given a ~~...~~ square matrix $A \in \mathbb{R}^{n \times n}$. Define the "characteristic polynomial of A" as

$$\Delta_A(s) = \det(sI - A) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots a_{n-1} s + a_n = 0$$

Thm 6.1 (Cayley - Hamilton)

Every matrix satisfies its own characteristic polynomial,

ie $\Delta_A(A) = A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1} A + a_n I = 0$

proof: (for A st. geometric mult = alg mult.)

$A = P \Lambda P^{-1}$ so

$$\Delta_A(A) = P\Lambda^n P^{-1} + a_1 P \Lambda^{n-1} P^{-1} + \cdots + a_{n-1} P \Lambda P^{-1} + a_n P P^{-1}$$

$$= P \underbrace{\left(\Lambda^n + a_1 \Lambda^{n-1} + \cdots + a_{n-1} \Lambda + a_n I \right)}_{} P^{-1}$$

diagonal matrix, where the i^{th} diagonal is

$$\lambda_i^n + a_1 \lambda_i^{n-1} + \cdots + a_{n-1} \lambda_i + a_n = 0$$

because λ_i is an eigenvalue of A

$\therefore \Lambda^n + a_1 \Lambda^{n-1} + \cdots + a_{n-1} \Lambda + a_n I = 0$

$\Rightarrow \Delta_A(A) = 0$

Given a polynomial $P(x)$ of degree p

and a polynomial $P_1(x)$ of degree $n < p$

Then we can write

$$P(x) = Q(x)\,P_1(x) + R(x)$$

where $Q(x)$ is of degree $p-n$

and $R(x)$ is of degree $n-1$ or less

Note

$$\frac{P(x)}{P_1(x)} = Q(x) + \frac{R(x)}{P_1(x)} \leftarrow \text{remainder}$$

Example

$$P(x) = x^3 + x^2 + x + 1$$

$$P_1(x) = x^2 + 2x + 1$$

$$\frac{P(x)}{P_1(x)} : \quad
\begin{array}{r|l}
 & x - 1 \\ \hline
x^2 + 2x + 1 & x^3 + x^2 + x + 1 \\
 & \underline{x^3 + 2x^2 + x} \\
 & \quad -x^2 \quad\quad + 1 \\
 & \quad \underline{-x^2 - 2x - 1} \\
 & \quad\quad\quad 2x
\end{array}$$

$$\therefore \quad \frac{P(x)}{P_1(x)} = \underbrace{x-1}_{Q(x)} + \frac{2x}{x^2+2x+1} \qquad R(x)$$

Given any matrix polynomial $P(A)$

$$e^A = I + A + \frac{1}{2!}A^2 + \cdots$$

$$\sin A = A - \frac{1}{3!}A^3 + \frac{1}{5!}A^5 + \cdots$$

$$\cos A = I - \frac{1}{2!}A^2 + \frac{1}{4!}A^4 + \cdots$$

Let $P_1(A) = \Delta_A(A)$ then

$$\underbrace{P(A)}_{\substack{\cancel{\text{order}}\ \infty \\ \text{degree}}} = \underbrace{Q(A)}_{\substack{\cancel{\text{order}}\ \infty \\ \text{degree}}} \underbrace{\Delta_A(A)}_{\substack{\cancel{\text{order}}\ n \\ \text{degree}}} + \underbrace{R(A)}_{\substack{\cancel{\text{order}}\ n-1 \\ \text{degree}}}$$

but since $\Delta_A(A) = 0$

$$\underbrace{P(A)}_{\substack{\cancel{\text{order}}\ \infty \\ \text{degree}}} = \underbrace{R(A)}_{\substack{\cancel{\text{order}}\ n-1 \\ \text{degree}}}$$

Thm Every infinite polynomial of A can be written as an $n-1$ degree polynomial

$$P(A) = \cancel{a_{n}}\ \cancel{a_{n}}\ \alpha_{n-1} A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I$$

Also note that since $\Delta_A(\lambda_i) = 0$ if $\lambda_i \in eig(A)$, then

$$P(\lambda_i) = Q(\lambda_i)\Delta_A(\lambda_i) + R(\lambda_i)$$

$$= R(\lambda_i)$$ ie the eigenvalues satisfy the same equation.

$\therefore$ if eigenvalues are unique, then we get n-equations in n-unknowns.

Example $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ $\lambda_1 = -2$, $\lambda_1 = -1$

Also we know that

$$e^{\lambda_1 t} = e^{-2t} = \alpha_1(-2) + \alpha_0$$

$$e^{\lambda_2 t} = e^{-t} = \alpha_1(-1) + \alpha_0$$

$$\Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} e^{-2t} \\ e^{-t} \end{pmatrix} = \frac{\begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}}{-2+1} \begin{pmatrix} e^{-2t} \\ e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} e^{-2t} \\ e^{-t} \end{pmatrix} = \begin{pmatrix} e^{-t} - e^{-2t} \\ 2e^{-t} - e^{-2t} \end{pmatrix}$$

$$\therefore e^{At} = \alpha_1 A + \alpha_0 I$$

$$= (e^{-t} - e^{-2t}) \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} + (2e^{-t} - e^{-2t}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}$$

Discrete ~~Ti~~ Time

Recall

$$\Phi(t, t_0) = \begin{cases} I & t = t_0 \\ A(t-1)A(t-2)\cdots A(t_0+1)A(t_0) & t > t_0 \end{cases}$$

Define $A^0 = I$, then

$$\Phi(t, t_0) = A^{t-t_0}$$

Methods for computing A^t

Z-transform: $A^t = Z^{-1}\{z(zI - A)^{-1}\}$

eigenvalues/eigenvectors $A^t = P\Lambda^t P^{-1}$

Cayley Hamilton (for $t \geq n$)