~~form~~

Since $\det(sI-A) = n$ is an n^{th} order polynomial, by the fundamental theorem of algebra there are n roots, or eigenvalues, some of which might be repeated.

Let $\det(sI-A) = (s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2}\cdots(s-\lambda_p)^{m_p}$

Def. ~~For eigenvalue and λ_i,~~
The <u>algebraic multiplicity</u> of λ_i is m_i

Def. the <u>geometric multiplicity</u> of λ_i is

$$q_i = \dim\{N(\lambda_i I - A)\}$$

<u>Example</u>: Double integrator) ~~from~~ $\ddot{y} = u$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

~~try~~ $\to \det(sI-A) = \det\begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix} = s^2$ ~~DLG~~
$= (s-0)^2$

$\Rightarrow \lambda_1 = 0$ has algebraic multiplicity $m_1 = 2$

$N\{(0I-A)\} = N\left\{\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\right\}$ or $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -x_2 = 0$

$\Rightarrow$ Null space spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \dim\{N(\lambda_1 I - A)\} = 1$

$\Rightarrow$ geometric multiplicity is $q_1 = 1$

~~Thus~~ In this case there are not two linearly independent eigenvectors and we can't diagonalize A with a similarity transformation.

To diagonalize, we instead find "generalized eigenvectors"

Given (λ_1, x_1) s.t. $Ax_1 = \lambda_1 x_1$, the generalized eigenvectors are found by solving the following chain:

$$
\begin{aligned}
Ax_2 &= \lambda_1 x_2 + x_1 \\
Ax_3 &= \lambda_1 x_3 + x_2 \\
&\vdots \\
A_{m_1} &= \lambda_1 x_{m_1} + x_{m_1 - 1}
\end{aligned}
\tag{2.1}
$$

Example: (double integrator)

$\lambda_1 = 0$, $x_1 = \begin{pmatrix}1\\0\end{pmatrix}$.

find x_2 by solving

$$Ax_2 = \lambda_1 x_2 + x_1$$

$$\Rightarrow \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}x_{21}\\x_{22}\end{pmatrix} = 0\begin{pmatrix}x_{21}\\x_{22}\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}$$

$$\Rightarrow x_{22} = 1$$

therefore ~~that~~ $x_2 = \begin{pmatrix}0\\1\end{pmatrix}$ is a generalized eigenvector of A.

Note that we can write (2.1) as

$$A[x_1 \; x_2 \cdots x_{m_1}] = [x_1 \cdots x_{m_1}] \underbrace{\begin{bmatrix} \lambda_1 & 1 & & 0 \\ & \lambda_1 & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_1 \end{bmatrix}}_{\text{Jordan block}}$$

It can be shown that there are $m_2 - 1$ linearly ~~indep~~ independent ~~eigenvect~~ generalized eigenvectors associated with each eigenvalue.

If $q_2 = 1$, then there is only one Jordan block with $m_2 - 1$ generalized eigenvectors generated by x_2.

However, if $1 < q_2 \le m_2$, then there are q_2 linearly independent eigenvectors, each of which can be used to generate generalized eigenvectors, and there are several potentially different ~~possible~~ possible Jordan blocks

For example, if $m_2 = 4$ and $q_2 = 2$, the possible Jordan blocks are

$$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix} \quad \text{and} \quad \lambda_1$$

$$\text{or} \quad \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$$

$$\text{or} \quad \lambda_1 \quad \text{and} \quad \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$$

which one is it?

To decide, generate the possible generalized eigenvectors for each eigenvalue, and pick the linearly independent ones.

Example

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

~~Sol~~ Since $\det(sI - A) = (\lambda - 1)^4$ we have $\lambda_1 = 1$, $m_1 = 4$

$$q_1 = \dim\left(N \begin{pmatrix} 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) = 2 \quad \text{since there are two l.i. rows}$$

So there are two linearly independent eigenvectors:

$$(\lambda_1 I - A) = \begin{pmatrix} 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_2 + x_3 - x_4 \\ -x_4 \\ -x_4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

so

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \text{ are eigenvectors}$$

Find possible generalized eigenvectors associated with x_1:

$$A x_3 = 1 \cdot x_3 + x_1 \quad \Rightarrow \quad (\lambda_1 I - A) x_3 = x_1$$

$$\Rightarrow \begin{pmatrix} 0 & -1 & 1 & -1 \\ & & 1 & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} -x_{32} + x_{33} - x_{34} \\ -x_{34} \\ -x_{34} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

could use $x_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$ which is l.i. of x_1 and x_2

try $A x_4 = \lambda_1 x_4 + x_3 \Rightarrow \begin{pmatrix} -x_2 + x_3 - x_4 \\ -x_4 \\ -x_4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \leftarrow$ impossible.

so try $A x_4 = \lambda_1 x_4 + x_2 \Rightarrow \begin{pmatrix} -x_2 + x_3 - x_4 \\ -x_4 \\ -x_4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$\therefore$ $x_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ works, so Jordan form is

$$A [x_1 \; x_3 \; x_2 \; x_4] = [x_1 \; x_3 \; x_2 \; x_4] \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{J}$$

For Jordan form we still have

$$A = P^{-1} J P$$

$$A^2 = AA = P^{-1} J P P^{-1} J P = P^{-1} J^2 P$$

$$\Rightarrow A^k = P^{-1} J^k P$$

$$\Rightarrow e^{At} = P^{-1} e^{Jt} P$$

If $J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_2 \end{pmatrix}$ where J_i is a Jordan block,

then $e^{At} = P^{-1} \begin{pmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_2 t} \end{pmatrix} P$

We need a formula for $e^{J_2 t}$

<u>Claim</u>: for

$$J_2 = \begin{pmatrix} \lambda_2 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 1 & 0 & \cdots \\ & & \ddots & \ddots & \\ & & & & \lambda_2 \end{pmatrix}$$ we have

$$e^{J_2 t} = e^{\lambda_2 t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \cdots & \frac{t^{n_2-1}}{(n_2-1)!} \\ 0 & 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_2-2}}{(n_2-2)!} \\ & & \ddots & & & \vdots \\ & & & & & t \\ & & & & & 1 \end{pmatrix}$$

$M(t)$

proof: we will show that

$$e^{J_i t}\Big|_{t=0} = I \qquad \text{and} \qquad \frac{d}{dt} e^{J_i t} = J_i e^{J_i t}$$

$$e^{J_i t}\Big|_{t=0} = e^{0}\begin{pmatrix} 1 & 0 & & 0 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = I$$

$$\frac{d}{dt} e^{J_i t} = \frac{d}{dt} e^{\lambda_i t} M(t) = \lambda_i e^{\lambda_i t} M(t) + e^{\lambda_i t} \frac{dM}{dt}$$

$$= \lambda_i e^{J_i t} + e^{\lambda_i t} \frac{dM}{dt}$$

where

$$\frac{dM}{dt} = \begin{pmatrix} 0 & 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ 0 & 0 & 1 & & & \frac{t^{n_i-3}}{(n_i-3)!} \\ & & & & & \vdots \\ & & & & & 1 \\ & & & & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ & 0 & & & 1 \\ & & & & 0 \end{pmatrix} M$$

$\Rightarrow$

$$\frac{d}{dt} e^{J_i t} = \lambda_i \cancel{M} e^{J_i t} + e^{\lambda_i t}\begin{pmatrix} 0 & 1 & & 0 \\ 0 & \cdots & & 0 \\ & & \ddots & \vdots \\ & 0 & & 0 \end{pmatrix} M$$

$=$