Lecture 8 Internal Stability

Matrix Norms

1. the one norm $\quad \|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}| \quad$ (max column sum)

2. the ∞-norm $\quad \|A\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}| \quad$ (max row sum)

3. the 2-norm $\quad \|A\|_2 = \sigma_{max}(A)$

4. the Frobenius norm $\quad \|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2} = \sqrt{\sum_{i=1}^{n} \sigma_i(A)^2}$

Thm: All matrix norms are equivalent i.e. $\exists$ a, b s.t.

$$a\|A\|_q \le \|A\|_p \le b\|A\|_q \quad \text{for any } p, q \ge 1$$

Submultiplicative property

$$\|AB\| \le \|A\| \, \|B\|$$

Subordinate or induced norms

$$\|A\|_p = \max_{x \ne 0} \frac{\|Ax\|_p}{\|x\|_p}$$

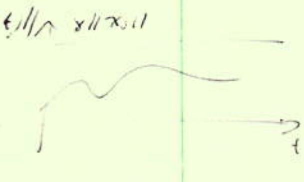
← matrix norm is induced by the vector norm.

Lyapunov Stability

Consider $\dot{x} = A(t)x + B(t)u$, $y = C(t)x + D(t)u$, $x(t_0) = x_0$ (2.1)

Def:

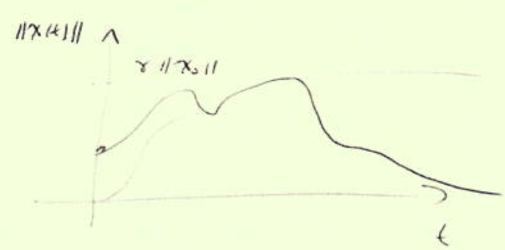
1. (2.1) is (marginally) stable in the sense of Lyapunov if, for every $x(t_0) = x_0$, $x(t) = \Phi(t, t_0)x_0$ is uniformly bounded ~~the~~



   i.e. $\|\Phi(t, t_0)\| \leq \gamma$ for some $\gamma < \infty$ and for all $t \geq t_0$

   [implies $\|x(t)\| \leq \|\Phi(t, t_0)\| \|x_0\| \leq \gamma \|x_0\|$]

2. Eq (2.1) is asymptotically stable if it is marginally stable and $x(t) \to 0$ as $t \to \infty$



3. Eq (2.1) is exponentially stable if $\exists\ c, \lambda > 0$ s.t.

$$\|x(t)\| \leq c e^{\lambda(t-t_0)} \|x(t_0)\| \quad \forall t \geq t_0$$

4. unstable if it is not marginally stable.

Eigenvalue Condition for stability (Time-Invariant case)

$$\dot{x} = Ax \quad , \quad x(0) = x_0 \tag{3.1}$$

Thm: System (3.1) is

1. marginally stable iff all eigenvalues of A have negative or zero real part and all Jordan blocks corresponding to eigenvalues with zero real part are 1×1

2. asymptotically stable iff all eigenvalues have strictly negative real parts

3. exponentially stable iff all eigenvalues have strictly negative real parts

4. unstable iff at least one eigenvalue has positive real part or zero real part with Jordan ~~b~~ block larger than 1×1

proof: exercise 8.3

Hint for non-Jordan form. $\cdot\ AP = P\Lambda \Rightarrow A = P\Lambda P^{-1}$

$$x(t) = P e^{\Lambda t} P^{-1} x_0$$

$$\Rightarrow \|x(t)\| \leq \|P\| \, \|e^{\Lambda t}\| \, \|P^{-1}\| \, \|x_0\|$$

Pick ~~the~~ norm where it is easiest to compute $\|e^{\Lambda t}\|$, for example ∞-norm is convenient

$$\|x(t)\|_\infty \leq \|P\|_\infty \|P^{-1}\|_\infty \max_{1\leq i\leq n} e^{\lambda_i t} \|x_0\| \qquad t \geq 0$$

$$= \cancel{\|P\|\|P^{-1}\|_\infty} c \, e^{\lambda t} \|x_0\|$$

where $c = \|P\|_\infty \|P^{-1}\|_\infty$ and $\lambda = \min_{1\leq i\leq n} Re\{\lambda_i\}$

Is it symmetric?

$$P^T = \left(\int_{t=0}^{\infty} e^{A^Tt} Q e^{At} dt\right)^T = \int_{t=0}^{\infty} (e^{A^Tt} Q e^{At})^T dt$$

$$= \int_0^{\infty} e^{A^Tt} Q e^{At} dt = P$$

Is P p.d.?

$$z^T P z = \int_0^{\infty} z^T e^{A^Tt} Q e^{At} z \, dt$$

let $w(t) = e^{At} z$ then

$$z^T P z = \int_0^{\infty} w^T(t) Q w(t) dt$$

but Q is ~~the~~ pd, therefore $w^T(t) Q w(t) > 0$

$$\Rightarrow z^T P z > 0$$

Also $z^T P z = 0$ when $z = 0$

Is soln unique?

Suppose $\bar{P}$ is another solution, then

$$A^T P + PA = -Q \quad \text{and} \quad A^T \bar{P} + \bar{P} A = -Q$$

$$\Rightarrow A^T(P - \bar{P}) + (P - \bar{P})A = 0$$

$$\Rightarrow e^{A^Tt} A^T (P - \bar{P}) e^{At} + e^{A^Tt} (P - \bar{P}) A) e^{At} = 0$$

$$\Rightarrow \frac{d}{dt} \left(e^{A^Tt} (P - \bar{P}) e^{At} \right) = 0$$

$\therefore$ $e^{A^Tt} (P - \bar{P}) e^{At}$ is constant

and since $e^{At} \to 0$ as $t \to \infty$, it must be that

$e^{A^Tt} (P - \bar{P}) e^{At} = 0$ but since $e^{At} \neq 0$ for finite t

we must have $P = \bar{P}$

(4.) $\Rightarrow$ (5.) is obvious

To show that (5.) $\Rightarrow$ (2.) we need

Thm (8.1) (Comparison Lemma)

Let $v(t)$ be a differentiable scalar signal s.t.

$$\dot{v}(t) \le \mu v(t) \quad \forall t \ge t_0$$

Then $$v(t) \le e^{\mu(t-t_0)} v(t_0)$$

proof: Let $u(t) = e^{-\mu(t-t_0)} v(t_0) \quad \forall t \ge t_0$

then $$\dot{u} = -\mu e^{-\mu(t-t_0)} v(t_0) + e^{-\mu(t-t_0)} \dot{v}$$

$$\le -\mu e^{-\mu(t-t_0)} v(t) + \mu e^{-\mu(t-t_0)} v(t) = 0$$

So $u(t)$ is non increasing

$$\Rightarrow u(t) = e^{-\mu(t-t_0)} v(t) \le u(t_0) = v(t_0)$$

$$\Rightarrow v(t) \le e^{\mu(t-t_0)} v(t_0)$$

(5.) $\Rightarrow$ (2.) Suppose $P = P^T > 0$ satisfies $A^T P + PA < 0$

Let $Q \triangleq -(A^T P + PA) > 0$

Let $v(t) = x^T P x \ge 0 \quad \forall t \ge 0$

then
$$\dot{v} = \dot{x}^T P x + x^T P \dot{x}$$
$$= x^T A^T P x + x^T P A x$$
$$= x^T (A^T P + PA) x$$
$$= -x^T Q x \le 0$$

$\therefore$ $v(t)$ is non increasing $\Rightarrow$ $v(t) = x^T(t) P x(t) \le v(0) = x^T(0) P x(0)$

Also since

$$\lambda_{min}(Q)\,\|x(t)\|^2 \le x^T(t)\,Q\,x(t) \le \lambda_{max}(Q)\,\|x(t)\|^2$$

we have

$$\dot{v} = -x^T Q x \le -\lambda_{min}(Q)\,\|x(t)\|^2$$

Also since

$$\lambda_{min}(P)\,\|x\|^2 \le v(t) = x^T P x \le \lambda_{max}(P)\,\|x\|^2$$

$$\Rightarrow \quad \dot{v} \le -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}\,v$$

$\therefore$ by the comparison lemma

$$v(t) \le e^{-\lambda(t-t_0)}\,v(t_0) \qquad \text{where } \lambda \triangleq \frac{\lambda_{min}(Q)}{\lambda_{max}(P)}$$

$$\Rightarrow \quad \|x(t)\|^2 \le \frac{\lambda_{max}(P)}{\lambda_{min}(P)}\,e^{-\lambda(t-t_0)}\,\|x(t_0)\|^2$$

Discrete time Stability

$$x(t+1) = A(t)x(t) + B(t)u(t), \quad x(t_0) \qquad (9.1)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

Def 8.2 Lyapunov Stability

1. System (9.1) is marginally stable if

$$\|\Phi(t, t_0)\| \le \gamma \qquad t \ge t_0$$

ie $\quad \|x(t)\| \le \gamma \|x_0\| \qquad t \ge t_0$

2. asymptotically stable if it is marginally stable and

$$\|x(t)\| \to 0 \quad \text{as } t \to \infty$$

3. exponentially stable if $\exists\, c > 0$, $\lambda < 1$ s.t.

$$\|x(t)\| \le c\lambda^{(t-t_0)} \|x(t_0)\|$$

4. unstable otherwise

Thm 8.3 (Eigenvalue Condition)

$$x^+ = Ax, \quad x(0) = x_0 \qquad (9.2)$$

is

1. Marginally stable iff all eigenvalues of A have magnitude ≤ 1 and Jordan block corresponding to eigenvalues with magnitude 1 are 1×1.

2. asymptotically + exponentially stable iff all eigenvalues of A have magnitude less than 1

3. unstable iff at least one eigenvalue has magnitude > 1 or magnitude $= 1$ and Jordan block larger than 1×1

Lyapunov stability

The following are equivalent:

1. System (9.2) is exponentially stable
2. All eigenvals of A have magnitude < 1
3. For every symmetric, p.d. Q, there exists a ~~__~~ unique $P = P^T > 0$ that satisfies

$$A^T P A - P = -Q$$

4. There exists a symmetric, p.d. ~~Q~~ P that satisfies

$$A^T P A - P < 0$$

proof: Very similar to continuous-time case, but use

$$P = \sum_{j=0}^{\infty} (A^T)^j Q A^j$$

and note that

$$\begin{aligned} A^T P A - P &= \sum_{j=0}^{\infty} (A^T)^{j+1} Q A^{j+1} - \sum_{j=0}^{\infty} (A^T)^j Q A^j \\ &= \sum_{j=1}^{\infty} (A^T)^j Q A^j - \sum_{j=0}^{\infty} (A^T)^j Q A^j \\ &= -(A^T)^0 Q A^0 \\ &= -Q \end{aligned}$$

Stability of Locally Linearized system

Given nonlinear system $\dot{x} = f(x)$

with equilibrium x^{eq} s.t. $f(x^{eq}) = 0$

Let $\delta x = x - x^{eq}$ and $A = \frac{\partial f}{\partial x}\Big|_{x = x^{eq}}$

then the linearized system is

$$\dot{\delta x} = A \delta x \tag{11.1}$$

Thm 8.5 Assume f is twice differentiable.

If system (11.1) is exponentially stable, then there exists a ball $B \subset \mathbb{R}^n$ around x^{eq} and constants $c, \lambda > 0$ s.t. ~~...~~ $x(t_0) \in B$ implies that

$$\|x(t) - x^{eq}\| \le c\, e^{-\lambda(t-t_0)} \|x(t_0) - x^{eq}\|$$

proof: From Taylor's thm, f - twice differentiable implies

$$f(x) = f(x^{eq}) + A\delta x + O(\|\delta x\|)^2$$

$\exists\, c, \bar{B}$

$$\Rightarrow \quad r(x) \triangleq f(x) - f(x^{eq}) - A\delta x \le c\|\delta x\|^2 \quad \forall x \in \bar{B}$$

For the linearized system, $\exists\, P = P^T > 0$ s.t.

$$A^T P + PA = -I$$

Let $v(t) = \delta x^T P \delta x$, then

$$\begin{aligned}
\dot{v} &= (\dot{x} - \dot{x}_{eq})^T P (x - x_{eq}) + (x - x^{eq})^T P (\dot{x} - \dot{x}_{eq}) \\
&= f^T P \delta x + \delta x^T P f \\
&= (A\delta x + r(x))^T P \delta x + \delta x^T P (A\delta x + r(x)) \\
&= \delta x^T (A^T P + PA) \delta x + r^T P \delta x + \delta x^T P r
\end{aligned}$$

$$\Rightarrow \quad \dot{v} \leq -\|\delta x\|^2 + 2\delta x^T P r$$

$$\leq -\|\delta x\|^2 + 2\|P\| \|\delta x\| \|r\|$$

~~If we can pick δ~~ δx so small enough to ensure that

$$-\|\delta x\|^2 + 2\|P\| \|\delta x\| \|r(x)\| \leq -\frac{1}{2} \|\delta x\|^2 \qquad (12.1)$$

then

$$\dot{v} \leq -\frac{1}{2}\|\delta x\|^2$$

but $\lambda_{min}(P)\|\delta x\|^2 \leq v(t) = \delta x^T P x \leq \lambda_{max}(P) \|\delta x\|^2$

$$\Rightarrow \quad -\frac{1}{2}\|\delta x\|^2 \leq -\frac{1}{2\lambda_{max}(P)} v$$

$$\Rightarrow \quad \dot{v} \leq -\frac{1}{2\lambda_{max}(P)} v$$

by the comparison lemma $v(t) \to 0$ exponentially fast

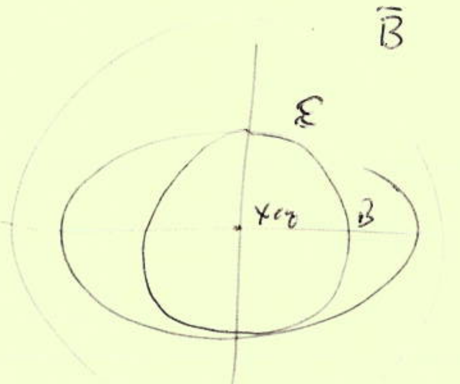
$\Rightarrow \delta x(t) \to 0$ exponentially fast.

So we need to ensure (12.1) is satisfied

$$-\|\delta x\|^2 + 2\|P\| \|\delta x\| \|r(x)\| \leq -\frac{1}{2}\|\delta x\|^2$$

Define $\mathcal{E} = \{x \in \mathbb{R}^n : (\delta x)^T P \delta x \leq \varepsilon\}$

and pick ε small enough so that $\mathcal{E} \subset \bar{B}$



Now for all $\delta x \in \mathcal{E}$ we have $\|r(x)\| \leq c\|\delta x\|^2$

so

$$-\|\delta x\|^2 + 2\|P\|\,\|\delta x\|\,\|r(x)\| \leq -(1 - 2c\|P\|\,\|\delta x\|)\,\|\delta x\|^2$$

Pick $B \subset \mathcal{E}$ so that

$$1 - 2c\,\|P\|\,\|\delta x\| \geq 1/2$$

ie $\|\delta x\| \leq \frac{1}{4c\|P\|}$ ie B has radius $\leq \frac{1}{4c\|P\|}$

$\therefore$ $\delta x \in B \Rightarrow$

$$-\|\delta x\|^2 + 2\|P\|\,\|\delta x\|\,\|r(x)\| \leq -1/2\,\|\delta x\|$$

~~We need to ensure that~~