Lecture 9 BIBO - Stability

Consider continuous-time system

$$\dot{x} = A(t)x + B(t)u \;, \quad y = C(t)x + D(t)u \quad x(0) = 0 \tag{1.1}$$

Solution - when $x(0) = 0$ is

$$y_f(t) = \int_0^t C(t)\Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t) \tag{1.2}$$

Def: The system is BIBO stable if there exists $g > 0$ s.t. for every input $u(t)$, the forced response satisfies

$$\sup_{t\in[0,\infty)} \|y_f(t)\| \le g \sup_{t\in[0,\infty)} \|u(t)\| \tag{1.3}$$

- Note: in the defn., initial conditions are assumed zero. If initial conditions are not zero, then only the forced response plays into the definition.

Thm 9.1 The following two statements are equivalent (ie iff)

1. The system (1.1) is BIBO stable
2. Every entry of $D(t)$ is uniformly bounded and for every entry $g_{ij}(t,\tau)$ of $C(t)\Phi(t,\tau)B(\tau)$ we have

$$\sup_{t\geq 0}\int_0^t |g_{ij}(t,\tau)|\,d\tau < \infty$$

proof:

~~(1.)⇒(2.)~~

(2.) ⇒ (1.)

$$y_f(t) = \int_0^t C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau + D(t)u(t)$$

$$\Rightarrow \|y_f(t)\| \leq \int_0^t \|C(t)\Phi(t,\tau)B(\tau)\|\|u(\tau)\|\,d\tau + \|D(t)\|\|u(t)\|$$

$$\leq \underbrace{\left(\int_0^t \|C(t)\Phi(t,\tau)B(\tau)\|\,d\tau + \sup_{t\geq 0}\|D(t)\|\right)}_{g}\sup_{t\geq 0}\|u(t)\|$$

For g to be finite we need

1) $\sup_{t\geq 0}\|D(t)\| < \infty$, ie every entry of $D(t)$ is uniformly bounded

2) $\int_0^t \|C(t)\Phi(t,\tau)B(\tau)\|\,d\tau < \infty$

But $\|A\| \leq \sum_{i,j}|a_{ij}|$ so (why?)

$$\int_0^t \|C(t)\Phi(t,\tau)B(\tau)\|\,d\tau \leq \sum_{i,j}\int_0^t |g_{ij}(t,\tau)|\,d\tau < \infty$$

$$\|A\| = \left\|\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \\ a & & & a \end{pmatrix}\right\| = \left\|\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & & \\ 0 & & 0\end{pmatrix} + \begin{pmatrix} 0 & a_{12} & 0 \\ \vdots & & \\ 0 & & \end{pmatrix} + \cdots + \begin{pmatrix} 0 & & 0 \\ & & \\ 0 & \cdots & a_{mn}\end{pmatrix}\right\| \leq \sum \left\|\begin{pmatrix} 0 & & \\ & a_{ij} & 0 \\ 0 & & 0\end{pmatrix}\right\|$$

$(1.) \Rightarrow (2.)$ which is the same as ~~+(2.)~~ $\neg(2.) \Rightarrow \neg(1.)$

Suppose (2.) is false because at least one ~~term~~ element of $D(t)$ is unbounded. Let $d_{ij}(t)$ be the unbounded element

Let $u_T(\tau) = \begin{cases} 0 & 0 \le \tau < T \\ e_j & \tau \ge T \end{cases}$

Note that $\sup_{t \ge 0} \|u_T(t)\| = 1$

at time T we have

$$y_f(T) = \int_0^T C(T)\Phi(T,\tau)u(\tau)\,d\tau + D(T)u_T(T) = D(T)u_T(T)$$

$$= D(T)e_j$$

$$\Rightarrow \sup_{t\in[0,\infty)} \|y_f(t)\| \ge \|y_f(T)\| = \|D(T)e_j\| = \left\| \begin{pmatrix} d_{ij}(T) \\ d_{nj}(T) \end{pmatrix} \right\| \ge |d_{ij}(T)|$$

Since at least one element ~~of~~ d_{ij} is unbounded, we can make

$\sup_{t\in[0,\infty)} \|y_f(t)\|$ arbitrarily large using a bounded input.

i.e. not BIBO stable

Now suppose (2) is false because $\sup_{t\ge0}\int_0^t |g_{ij}(t)|\,dt$ is unbounded for some i,j

Let $u_T(\tau) = \begin{cases} +e_j, & g_{ij}(t,\tau) \ge 0 \\ -e_j, & g_{ij}(t,\tau) \le 0 \end{cases} \quad \tau \ge 0$

$$\Rightarrow y_f(t) = \begin{pmatrix} \int_0^t |g_{ij}(t,\tau)|\,d\tau \\ \vdots \\ \int_0^t |g_{nj}(t,\tau)|\,d\tau \end{pmatrix} \pm \begin{pmatrix} d_{ij}(t) \\ \vdots \\ d_{nj}(t) \end{pmatrix}$$

Note that $\sup_{t\ge0} \|u_T(t)\| = 1$

$$\sup_{t\ge0} \|y_f(t)\| \ge \|y_f(T)\| \ge \underbrace{\int_0^T |g_{ij}(T,\tau)|\,d\tau}_{\text{can make this arbitrarily large}} \pm d_{ij}(t)$$

$\Rightarrow \sup \|y_f(t)\|$ not bounded

Thm 9.2

For the time-invariant system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

the following two statements are equivalent:

1.) the system is uniformly BIBO stable

2) for every entry $\bar{g}_{ij}(\rho)$ of $Ce^{A\rho}B$ we have

$$\int_0^\infty |\bar{g}_{ij}(\rho)|\, d\rho < \infty$$

Frequency Domain Characterization

Thm 9.3

For the linear time-invariant system

$$\dot{x} = Ax + Bu \qquad y = Cx + Du$$

where $\hat{G}(s) = \mathcal{L}\{C(sI-A)^{-1}B\}$

the following statements are equivalent:

1.) The system is uniformly BIBO stable

2.) Every pole of every entry in $\hat{G}(s)$ has a strictly negative real part.

proof: we simply need to establish that

$$\int_0^\infty |g_{ij}(\rho)|\, d\sigma < \infty \iff \text{the poles of } \hat{g}_{ij}(s) \text{ are in open LHP}$$

we know that all elements of $\hat{G}(s) = \mathcal{L}\{C(sI-A)^{-1}B\}$ are strictly proper, rational transfer functions

$\therefore$ fewer zeros than poles

$$\Rightarrow \quad \hat{g}_{ij}(s) = \frac{a_0 s^q + a_1 s^{q-1} + \cdots + a_{q-1} s + a_q}{(s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2} \cdots (s-\lambda_k)^{m_k}} \qquad \sum m_k > q$$

$$= \frac{a_{11}}{s-\lambda_1} + \frac{a_{12}}{(s-\lambda_1)^2} + \cdots + \frac{a_{1m_1}}{(s-\lambda_1)^{m_1}} + \cdots$$

$$+ \frac{a_{k1}}{s-\lambda_k} + \frac{a_{k2}}{(s-\lambda_k)^2} + \cdots + \frac{a_{km_k}}{(s-\lambda_k)^{m_k}}$$

The inverse Laplace transform is

$$g_{ij}(t) = a_{11} e^{\lambda_1 t} + a_{12} t e^{\lambda_1 t} + \cdots + a_{1m_1} t^{m_1 - 1} e^{\lambda_1 t} + \cdots$$

$$+ a_{k1} e^{\lambda_k t} + \cdots + a_{km_k} t^{m_k - 1} e^{\lambda_k t}$$

for each term we have

$$\int_0^\infty a_{ij} t^{(j-1)} e^{\lambda_i t} \, dt < \infty \quad \text{iff} \quad \mathrm{Re}\{\lambda_i\} < 0$$

BIBO vs Lyapunov Stability

Thm 9.4

[exponential stability i.s.L. in the sense of Lyapunov]

$\Rightarrow$ [BIBO stability]

proof: exponential stability $\Rightarrow$ eigenvalues of A are in open RHP

$\Rightarrow$ all poles of $C(sI-A)^{-1}B$ are in open RHP

$\Rightarrow$ BIBO stable

However the converse is not true because there may be pole-zero cancelations in $C(sI-A)^{-1}B$

Example $\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = [1 \ \ 1] x$

clearly $eig(A) = 1, -2$ $\therefore$ not exponentially stable i.s.L.

However

$$C(sI-A)^{-1}B = (1 \ \ 1)\begin{pmatrix} s-1 & 0 \\ 0 & s+2 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= (1 \ \ 1)\frac{\begin{pmatrix} s+2 & 0 \\ 0 & s-1 \end{pmatrix}}{(s+2)(s-1)}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= (1 \ \ 1)\begin{pmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s+2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= (1 \ \ 1)\begin{pmatrix} 0 \\ \frac{1}{s+2} \end{pmatrix}$$

$$= \frac{1}{s+2} \Rightarrow \text{BIBO stable}$$

Discrete - Time

$$x(t+1) = A(t)\,x(t) + B(t)u(t)\ , \quad y(t) = C(t)x(t) + D(t)u(t) \qquad (7.1)$$

$$\rightarrow \quad y_f(t) = \sum_{\tau=0}^{t-1} C(t)\,\Phi(t,\tau+1)\,B(\tau)u(\tau) + D(t)\,u(t)$$

$$t \geq 0$$

Def System (7.1) is BIBO stable if there exists a constant $g > 0$ s.t. for every input, the forced response satisfies

$$\sup_{t\in\mathbb{N}} \|y_f(t)\| \leq g \sup_{t\in\mathbb{N}} \|u(t)\|$$

Thm 9.5 The following two statements are equivalent:

1) (7.1) is BIBO stable

2) Every entry of $D(t)$ is uniformly bounded and

$$\sup_{t\geq 0} \sum_{\tau=0}^{t-1} |g_{ij}(t,\tau)| < \infty$$

where $G(t,\tau) = C\,\Phi(t,\tau)\,B(\tau)$

Thm 9.6 The following three statements are ~~equal~~ equivalent:

1.) The time-invariant system

$$x^+ = Ax + Bu\ , \quad y = Cx + Du$$

is BIBO stable

2.) Every entry of $G(p) = CA^pB$ satisfies

$$\sum_{p=1}^{\infty} |g_{ij}(p)| < \infty$$

3) Every pole of $\hat{G}(z) = Z\{CA^pB\}$ ~~satisfies~~ has magnitude in the open unit circle