Mathematical Preliminaries

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$y = (y_1, \dots, y_n) \in \mathbb{R}^n$

The inner product in $\mathbb{R}^n$ is defined as

$$x^T y = \sum_{i=1}^{n} x_i y_i$$

Norms The norm of a vector has the following properties

1) $\|x\| \geq 0 \quad \forall x \in \mathbb{R}^n \quad \|x\| = 0$ iff $x = 0$
2) $\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality)
3) $\|\alpha x\| = |\alpha| \, \|x\| \quad \forall \alpha \in \mathbb{R}, \; x \in \mathbb{R}^n$

p-norms on $\mathbb{R}^n$ are defined as follows:

$$\|x\|_p = \left(|x_1|^p + \dots + |x_n|^p\right)^{1/p} \qquad 1 \leq p \leq \infty$$

$$\cancel{\|x\|_\infty = \max_i |x_i|}$$

Note that

$$\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$$

$$\|x\|_2 = \sqrt{x_1^2 + \dots + x_n^2} = \sqrt{x^T x}$$

$$\|x\|_\infty = \max_i |x_i|$$

Hölder's inequality:

$$|x^T y| \leq \|x\|_p \|y\|_q \quad , \; \frac{1}{p} + \frac{1}{q} = 1$$

This is a generalization of the Cauchy-Schwartz inequality

$$|x^T y| \leq \|x\|_2 \|y\|_2$$

Matrix norms:

The <u>induced</u> p-norm of $A \in \mathbb{R}^{m \times n}$ is given by

$$\|A_p\| = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p = 1} \|Ax\|_p$$

<u>Convergence of Sequences</u>

The following statements are equivalent

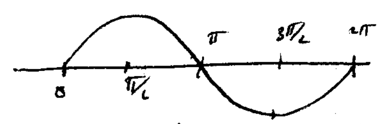
(1) the sequence $\{x_k\}$ converges to x

(2) $\|x_k - x\| \to 0$ as $k \to \infty$

(3) $\forall \varepsilon > 0$, $\exists N$ s.t. $\|x_k - x\| < \varepsilon$ $\forall k \geq N$.

x is an accumulation point of $\{x_k\}$ if there is a ~~sequ~~ subsequence of $\{x_k\}$ that converges to x.

Example : $x_k = \sin(k\pi/2)$



This sequence does not converge, but it has five accumulation points.

Fact: A bounded sequence has at least one accumulation point.

A sequence $\{x_k\}$ is (strictly) monotonically increasing if
$x_k \; (<) \leq x_{k+1}$ $\forall k$.

Fact: An increasing sequence that is bounded from above converges.

Similar statements can be made for decreasing sequences.

<u>Sets</u>: $S \subset \mathbb{R}^n$ is <u>open</u> if $\forall x \in \mathbb{R}^n$, $\exists \varepsilon$ such that $N(x, \varepsilon) = \{z \in \mathbb{R}^n : \|z - x\| < \varepsilon\} \subset S$.

S is <u>closed</u> in $\mathbb{R}^n$ if $\mathbb{R}^n \setminus S$ is open.

Fact: S is closed iff every convergent sequence $\{x_k\}$ in S converge to an element in S.

must contain boundary point to be closed.

A set $S \subset \mathbb{R}^n$ is <u>bounded</u> if $\exists r > 0$ such that $\|x\| \le r \quad \forall x \in \mathbb{R}^n$.

In $\mathbb{R}^n$, a set S is called <u>compact</u> if it is closed and bounded.

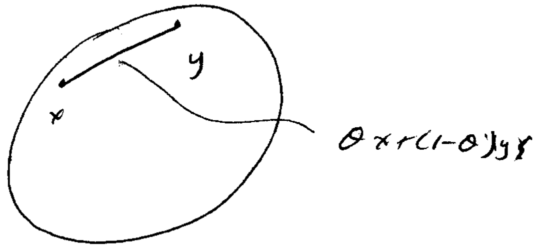
A point p is a <u>boundary point</u> of S if in <u>every</u> neighbo-hood of p, there is a point in S and a point not in S.

$\partial S = \{x \in \mathbb{R}^n : x \text{ is a boundary point of } S\}$

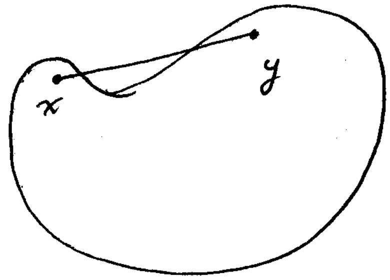
Fact: a closed set contains all its boundary points

Fact an open set contains none of its boundary points.

A set $S \subset \mathbb{R}^n$ is <u>convex</u> if $\forall x, y \in S$ and $0 \le \theta \le 1$, $\theta x + (1-\theta) y \in S$



Convex



Not convex

__Continuity__ Let $f : S_1 \to S_2$.

The following statements are equivalent:

(1) f is continuous at $x \in S_1$

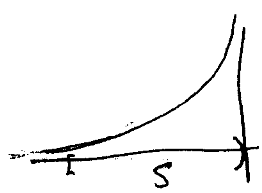
(2) $x_k \to x \Rightarrow f(x_k) \to f(x)$

(3) $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $\|x-y\| < \delta \Rightarrow \|f(x)-f(y)\| < \varepsilon$.

f is continuous on S_1 if it is continuous at each $x \in S_1$.

f is uniformly continuous on S_1 if, ~~[illegible]~~ given ε, the same δ works for all $x \in S_1$

Example



this function is continuous on S but not uniformly cont.

If S is compact, then ~~[illegible]~~ continuity and uniform continuity are equivalent.

If f_1 and f_2 are continuous then

$\alpha_1 f_1 + \alpha_2 f_2$ and $f_2(f_1)$ are cont.

Let $S \subset \mathbb{R}^n$ and $f: S \to \mathbb{R}^m$ ~~then~~ and let

$$f(S) = \{ y \in \mathbb{R}^m : y = f(x) \text{ for some } x \in S \}$$

$f(S)$ is called the <u>image</u> of f.

Fact: if S is compact and f is continuous then $f(S)$ is compact.

The function f is called <u>one-to-one</u> if

$$x \neq y \Rightarrow f(x) \neq f(y)$$

Fact: if (1) f - continuous

(2) f - one-to-one

(3) f is compact

then f has a continuous inverse f^{-1} on $f(S)$ such that $f^{-1}(f(x)) = x$.

~~[illegible]~~

$f: \mathbb{R} \to \mathbb{R}^n$ is piecewise cont. if it is continuous except at a <u>finite</u> number of points, and if at those points the ~~right side~~s right and left hand limits exist.

Differentiable functions:

$f:\mathbb{R}\to\mathbb{R}$ is *differentiable* at x if

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad \text{exists.}$$

$f:\mathbb{R}^n\to\mathbb{R}^m$ is continuously differentiable ~~if~~ at x_0 if

$\frac{\partial f_i}{\partial x_j}$ exist and is continuous at x_0

~~in which case we write~~ If $f:\mathbb{R}^n\to\mathbb{R}$ then

$$\frac{\partial f}{\partial x} = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right)$$

If $f:\mathbb{R}^n\to\mathbb{R}^m$ then

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

If $h(x) = g(f(x))$ then $\left.\frac{\partial h}{\partial x}\right|_{x=x_0} = \left(\left.\frac{\partial g}{\partial f}\right|_{f=f(x_0)}\right)\left(\left.\frac{\partial f}{\partial x}\right|_{x=x_0}\right)$

If $x, y \in \mathbb{R}^n$ define $L(x,y) = \{ x \in \mathbb{R}^n : z = \theta x + (1-\theta) y \ , \ 0 < \theta < 1 \}$.

Mean Value Theorem

Assume $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable at each $x \in S \subset \mathbb{R}^n$. Let $x, y \in S$ s.t. $L(x,y) \subset S$.

Then there exists a point $z \in L(x,y)$ s.t.

$$f(y) - f(x) = \left.\frac{\partial f}{\partial x}\right|_{x=z} (y - x) \ .$$

If $f : \mathbb{R} \to \mathbb{R}$ then $\left.\frac{\partial f}{\partial x}\right|_{x=z} = \underbrace{\frac{f(y) - f(x)}{y - x}}_{\text{rise over run,}}$

is slope of