In this class we will model dynamic systems using state space models of the form

$$\dot{x}_1 = f_1(t, x_1, x_2, \ldots, x_n, u_1, \ldots, u_p)$$
$$\vdots$$
$$\dot{x}_n = f_n(t, x_1, x_2, \ldots, x_n, u_1, \ldots, u_p)$$
$$y_1 = h_1(t, x_1, \ldots, x_n, u_1, \ldots u_p)$$
$$\vdots$$
$$y_q = h_q(t, x_1, \ldots x_n, u_1, \ldots u_p)$$

Letting $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $u = \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix}$ $y = \begin{pmatrix} y_1 \\ \vdots \\ y_q \end{pmatrix}$

we can write

$$\dot{x} = f(t, x, u)$$
$$y = h(t, x, u)$$

The first part of the book will deal with unforced equations of the form

$$\dot{x} = f(t, x)$$
$$y = h(t, x)$$

In the time-invariant case we get

$$\dot{x} = f(x)$$
$$y = h(x)$$

which has traditionally been called an "autonomous" system.

<u>Def</u>: A point ~~of~~ x^* is called an <u>equilibrium point</u> if $f(x^*) = 0$. ~~which~~

Equilibrium points have the property that

if $x_0 = x^*$ then $x(t) = x^*$ $\forall t \geq 0$

Note that $\dot{x} = f(x^*) = 0$ implies that x never changes.

Compare

$$\dot{x} = f(t, x)$$
$$y = h(t, x)$$

with the linear state-space model

$$\dot{x} = A(t)x + B(t)u$$
$$y = C(t)x + D(t)u$$

Recall that Laplace + Fourier transforms are useful because ~~the~~ convolution in time domain transforms to multiplication in frequency domain, and convolution completely characterizes input-output behavior for LTI systems.

For nonlinear systems, convolution no longer characterizes input-output behavior $\therefore$ transform methods are much less useful. Almost all analysis is carried out in time-domain

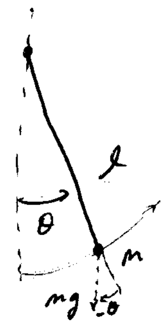
Nonlinear systems have much richer behavior than linear systems.

Some "nonlinear phenomena" that do not occur in linear systems include

- Finite escape time
- Multiple isolated equilibria
- Limit cycles
- chaos
- Bifurcation

Chap 1 introduces a number of example of nonlinear systems. These examples will be used for illustration throughout the ~~book~~ book.

Example: Pendulum



Newtons 2nd law

$$J\ddot{\theta} = \Sigma \text{ torques}$$

where $J = \int a^2 dm = ml^2$

$$ml^2\ddot{\theta} = \Sigma \text{ torques}$$

torque due to gravity is $\tau_g = r \times F$
$= |r||F| \sin\theta$
$= l m g \sin\theta$

adding viscous friction $\tau_v = \underbrace{-k}\, l\dot{\theta}$

opposing force proportional to velocity

we get

$$ml^2\ddot{\theta} = -mgl\sin\theta - kl\dot{\theta}$$

If we apply a torque T at the hub then the equations of motion are

$$ml^2\ddot{\theta} = -mgl\sin\theta - kl\dot{\theta} + T$$

Setting $x_1 = \theta$, $x_2 = \dot{\theta}$, $u = T$, we get

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin x_1 - \frac{k}{m\ell} x_2 + \frac{1}{m\ell^2} u \end{aligned}\right\} f(x,u)$$

The unforced equilibrium points are found by setting $f(x,0) = 0$

ie

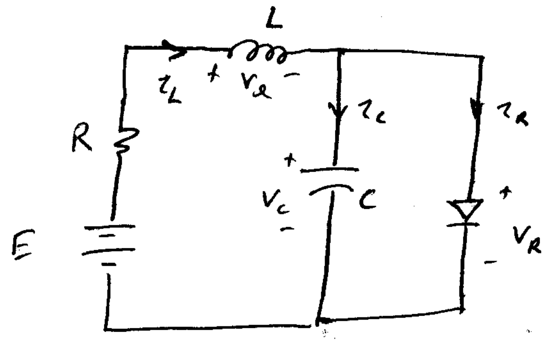
$$x_2 = 0$$

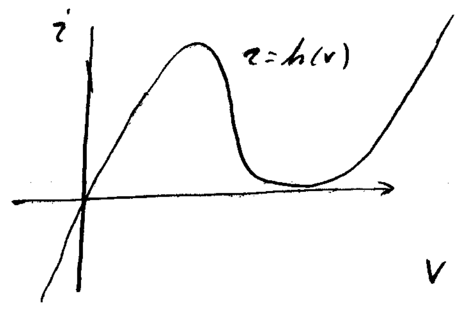
$$-\frac{g}{\ell}\sin x_1 - \frac{k}{m\ell} x_2 = 0$$

The equilibrium points are therefore $x = \begin{pmatrix} n\pi \\ 0 \end{pmatrix}$

$n = 0, \pm 1, \ldots$

Example — Tunnel Diode Circuit





diode characteristics

$$i_C = C \frac{dv_C}{dt}$$

$$V_L = L \frac{di_L}{dt}$$

Kirchhoff's current law

$$i_C + i_R - i_L = 0$$

Kirchhoff's voltage law

$$v_C - E + R i_L + V_L = 0$$

Define the state variables as

$$x_1 = v_C \quad , \quad x_2 = i_L \quad , \quad u = E$$

which gives

$$x_1 - u + R x_2 + L \frac{dx_2}{dt} = 0$$

$$C \frac{dx_1}{dt} + h(x_1) - x_2 = 0$$

or

$$\left.\begin{aligned} \dot{x}_1 &= \frac{1}{C} x_2 - \frac{1}{C} h(x) \\ \dot{x}_2 &= -\frac{1}{L} x_1 - \frac{R}{L} x_2 + \frac{1}{L} u \end{aligned}\right\} \triangleq f(x, u)$$

The ~~unforced~~ equilibrium points are found by setting

$$f(x,0) = 0$$

ie

$$x_2 = h(x_1)$$

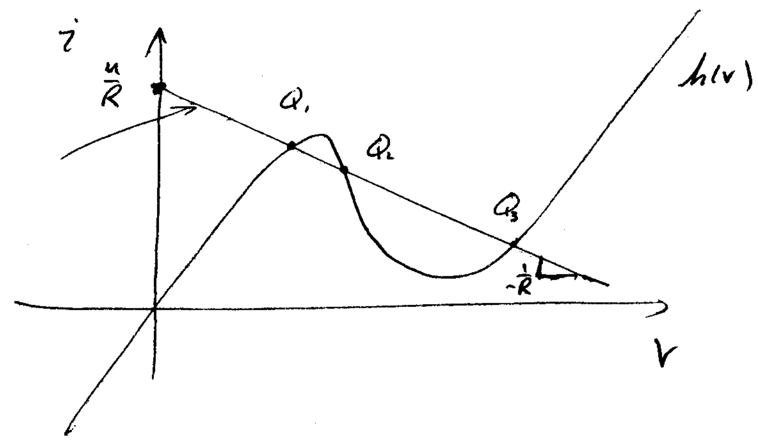
$$x_1 + R x_2 = u$$

or

$$x_1 + R\,h(x_1) = u$$

or

$$h(x_1) = \frac{1}{R} u - \frac{1}{R} x_1$$

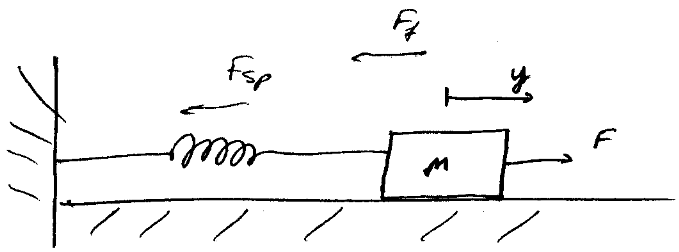


Notice that for various values of u, the system may have 1, 2, or 3 equilibrium points.

∴ The very structure of the system changes.

Changes from one equilibrium point to three is called a bifurcation.

Example Mass - Spring System



$y \triangleq$ position from zero spring force

$F_{sp} \triangleq$ spring force

$F_f \triangleq$ force due to friction

$F \triangleq$ applied force

From Newton's 2^{nd} law

$$m\ddot{y} = F - F_{sp} - F_f$$

Assume spring force is only due to displacement

ie $\quad F_{sp} = g(y)$

A linear spring is described by Hooke's law as

$$g(y) = k_{sp}\, y$$

but springs are rarely linear. Common models that describe 2^{nd} order effects are

softening springs: $\quad g(y) = k(1 - a^2y^2)\, y \qquad |ay| < 1$

hardening spring: $\quad g(y) = \cancel{k(1+a^2k} \quad k(1 + a^2y^2)\, y$

There are several friction effects:

1) Viscous friction:
   - due to air (or fluid) resistance
   - proportional to velocity, i.e. $F_r = c\dot{y}$

2) Kinetic friction:
   - due to surface tension
   - $F_k = \mu_k mg$ opposes motion

3) static friction
   - due to surface tension
   - F_s - acts parallel to surface + is bounded by $\mu_s mg$ $0 < \mu_s < 1$

Kinetic + Static friction can be combined to produce dry friction

$$F_d = \begin{cases} -\mu_k mg & \text{if } \dot{y} < 0 \\ F_s & \dot{y} = 0 \\ \mu_k mg & \text{if } \dot{y} > 0 \end{cases}$$

If the spring force is linear we have

$$m\ddot{y} + ky + c\dot{y} + \eta(y, \dot{y}) = 0$$

where

$$\eta(y,\dot{y}) = \begin{cases} \mu_k mg \operatorname{sign}(\dot{y}) & |\dot{y}| > 0 \\ -ky & \dot{y} = 0 \quad |y| \leq \mu_s mg/k \\ -\mu_s mg \operatorname{sign}(y) & \dot{y} = 0 \quad |y| \geq \mu_s mg/k \end{cases}$$

Setting $x_1 = y$, $x_2 = \dot{y}$ the state space model is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 - \frac{1}{m}\eta(x_1, x_2)$$

Note: Many equilibria, $f(x)$ is discontinuous.

However note that when

$x_2 > 0$ we can write

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m} x_1 - \frac{c}{m} x_2 - \mu_k g$$

and when $x_2 < 0$ we can write

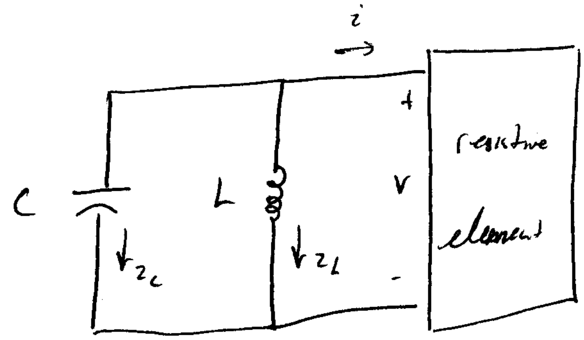
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m} x_1 - \frac{c}{m} x_2 + \mu_k g$$

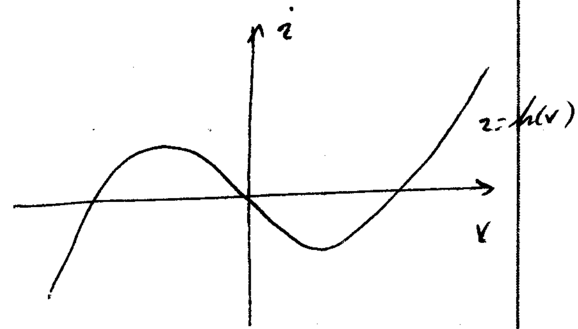
which are both linear (or affine).

Therefore the system is piecewise linear.

Example: Negative Resistance Oscillator

Consider the circuit





Resistive Element

We can write

$$i_C + i_L + i = 0$$

or

$$C\frac{dv}{dt} + \frac{1}{L}\int_{-\infty}^{t} v(s)\,ds + h(v) = 0$$

or

$$LC\frac{d^2v}{dt} + v + L\,h'(v)\frac{dv}{dt} = 0 \qquad \text{where } h'(v) = \frac{\partial h}{\partial v}(v)$$

Changing the time scale to $\tau = \frac{t}{\sqrt{CL}} \Rightarrow d\tau = \frac{dt}{\sqrt{LC}}$

$\Rightarrow (d\tau)^2 = d\tau(d\tau) = \frac{dt^2}{LC}$

~~let~~ and defining $\dot{v} = \frac{dv}{d\tau}$ gives

$$\ddot{v} + \varepsilon h'(v)\dot{v} + v = 0 \qquad \text{where } \varepsilon = \sqrt{L/C}.$$

Setting $x_1 = v$ and $x_2 = \dot{v}$ we get

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - \varepsilon h'(x_1)x_2$$

We could have also chosen the state variables as $z_1 = i_L \qquad z_2 = v_c$

then

$$\frac{d\hat{z}_1}{dt} = \frac{d\hat{i}_L}{dt} = \frac{1}{L}\frac{dv_c}{dt} = \frac{1}{L} z_2$$

$$\frac{d\hat{z}_2}{dt} = \frac{dv}{dt} = -\frac{1}{c} i_L - \frac{1}{c} h(v) = -\frac{1}{c}\left[z_1 + h(z_2)\right]$$

again letting $\tau = \frac{t}{\sqrt{LC}}$ gives

$$\dot{z}_1 = \frac{1}{\varepsilon} z_2$$

$$\dot{z}_2 = -\varepsilon\left[z_1 + h(z_2)\right]$$

While the state variables are different, we are representing the same physical system, therefore we should be able to transform between the two representations in a smooth way

Note that

$$\begin{aligned} z_1 = i_L &= -i - i_c \\ &= -h(v) - c\frac{dv}{dt} \\ &= -h(x_1) - \sqrt{\tfrac{c}{L}}\frac{dv}{d\tau} \\ &= -h(x_1) - \tfrac{1}{\varepsilon} x_2 \end{aligned}$$

$$z_2 = v_c = x_1$$

$$\therefore z = T(x) = \begin{pmatrix} -h(x_1) - \frac{1}{\varepsilon} x_2 \\ x_1 \end{pmatrix}$$

Also

$$x_1 = v = z_2$$

$$x_2 = \frac{dv}{d\tau} = \sqrt{LC}\frac{dv}{dt} = \sqrt{LC}\left(-\frac{1}{c} i_L - \frac{1}{c} h(v)\right)$$

$$= -\sqrt{\tfrac{L}{c}}\, z_1 - \sqrt{\tfrac{L}{c}}\, h(z_2)$$

$$\Rightarrow \quad x = T^{-1}(z) = \begin{pmatrix} z_2 \\ -\varepsilon z_1 - \varepsilon h(z_2) \end{pmatrix}$$

T is a continuous transformation with a continuous inverse. Such transformations are called diffeomorphisms and play an important role in nonlinear analysis (similar to similarity transformations in linear systems theory).