Chap 13 Feedback Linearization

Consider the pendulum

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a\left[\sin(x_1+\delta) - \sin\delta\right] - b x_2 + c u$$

Note that all of the nonlinearities are in the same channel as the input.

Therefore picking the input as

$$u = \frac{a}{c}\left[\sin(x_1+\delta) - \sin\delta\right] + \frac{v}{c}$$

results in the linearized system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -b x_2 + v$$

Note: we did not take Jacobians, or assume small signals! We used the feedback signal to cancel the nonlinearity.

We can now use our favorite linear control technique to stabilize the transformed system.

For example, if we select

$$v = -k_1 x_1 - k_2 x_2$$

then the final control signal is

$$u = \frac{a}{c}\left[\sin(x_1+\delta) - \sin\delta\right] - \frac{k_1}{c} x_1 - \frac{k_2}{c} x_2$$

Question: Is it always possible to use the feedback signal to cancel the nonlinearities?

Answer: No! We will Develop explicit conditions later in the chapter

Question: Are there draw ~~p~~ backs ~~to~~ to feedback linearization, even when it is possible?

Answer: Yes! Sometimes the nonlinearity helps stabilize the system. Canceling a good, or helpful nonlinearity ~~takes~~ wastes energy.

Example:

$$\dot{x} = -x^3 + u$$

The system is stable with $u = 0$

The feedback linearization technique requires

$$u = x^3 + v \quad \text{to get}$$

$$\dot{x} = v$$

we then pick v to stabilize the system, eg, $v = -\alpha x$,

then $\dot{x} = -x^3 + \underset{\substack{\uparrow \\ \text{very expensive for large } x.}}{(x^3 - \alpha x)} = -\alpha x$

What if the nonlinearity is not in the input channel?

Example:
$$\dot{x}_1 = a \sin x_2$$
$$\dot{x}_2 = -x_1^2 + u$$

we cannot chose u to cancel $a \sin x_2$

The essential idea is to chose a change of coordinates that moves all of the nonlinearities into the input channel.

For example, let

$$z_1 = x_1$$
$$z_2 = a \sin x_2 = \dot{x}_1$$

then
$$\dot z_1 = \dot x_1 = a \sin x_2 = z_2$$
$$\dot z_2 = a \cos x_2 \, \dot x_2 = a \cos x_2 \, (-x_1^2 + u)$$

Letting $u = x_1^2 + \frac{1}{a \cos x_2} v$ gives

$$\dot z_1 = z_2$$
$$\dot z_2 = v$$

we can now choose v to stabilize the z-system
eg $v = -k_1 z_1 - k_2 z_2 \qquad k_1, k_2 > 0$

$\therefore$ In the original coordinates we have
$$u = x_1^2 + \frac{1}{a \cos x_2}(-k_1 z_1 - k_2 z_2)$$
$$= x_1^2 + \frac{1}{a \cos x_2}\left(-k_1 x_1 - k_2 \, a \sin(x_2)\right)$$

This trick guarantees that $(z_1, z_2) \to 0$
In our example $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ a \sin x_2 \end{pmatrix} \to 0 \iff \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} 0 \\ k\pi \end{pmatrix}$ k-integer

If that satisfies the objective, then we are done.

If the objective is tracking, then feedback linearization can actually make life harder:

<u>Example</u>:
$$\dot x_1 = a \sin x_2$$
$$\dot x_2 = -x_2^2 + u$$
$$y = x_2$$
$\leftarrow$ the ~~ofte~~ goal may be to force y to track some signal.

~~Using feedback~~

Transforming the system using
$$z_1 = x_1$$
$$z_2 = a\sin x_2$$
$$u = x_1^2 + \frac{1}{a\cos x_2} v$$

gives
$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = v$$
$$y = \sin^{-1}\left(\frac{z_2}{a}\right)$$

we transformed a linear output equation into a nonlinear output equation.

Instead of linearizing the state equations, for tracking, it is often better to linearize the input-output map.

Example
$$\dot{x}_1 = a\sin x_2$$
$$\dot{x}_2 = -x_2^2 + u$$
$$y = x_2$$

$$\Rightarrow \quad \dot{y} = \dot{x}_2 = -x_2^2 + u$$

Now letting $u = x_2^2 + v$ gives

$$\dot{y} = v$$

or

$$\dot{x}_1 = a\sin x_2 \quad \} \text{ zero dynamics.}$$

$$\left.\begin{array}{l}\dot{x}_2 = v \\ y = x_2\end{array}\right\} \text{ input-output dynamics.}$$

Note that x_1 does not affect the output y. However x_2 affects x_1 which may be unstable if v is not chosen correctly.

~~For example if~~

Input - Output Linearization

Input-output linearization is the study of linearizing the input-output map.

Relative degree

Recall that for a linear system

$$Y = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} u$$

the relative degree is defined as $\rho = n - m$, i.e., the pole-zero excess.

Recall that a transfer function

$$Y = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} u$$

has a state space representation given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [b_0 \; b_1 \; \cdots \; b_m \; 0 \cdots 0] x$$

Note that:

$$\dot{y} = b_0 \dot{x}_1 + b_1 \dot{x}_2 + \cdots b_m \dot{x}_m$$

$$= b_0 x_2 + b_1 x_3 + \cdots b_m x_{m+1} \quad \leftarrow \text{no } u$$

$$\ddot{y} = b_0 x_3 + b_1 x_4 + \cdots + b_m x_{m+2} \quad \leftarrow \text{no } u$$

$\vdots$

$$y^{(n-m-1)} = b_0 x_{n-m} + \cdots + b_m x_n$$

$$y^{(n-m)} = b_0 x_{n-m+1} + \cdots + b_m \dot{x}_n$$

$$= \underbrace{b_0 x_{n-m+1} + \cdots + b_m(-a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n)}_{\text{States}} + b_m u$$

u shows up.

The input does not show up in derivative of the output until $\rho = n-m$ differentiation.

Therefore, while nonlinear systems do not have poles and zeros, we can still define the notion of relative degree.

First some notation:

Let $W: \mathbb{R}^n \to \mathbb{R}$ be a scalar function of x and $q: \mathbb{R}^n \to \mathbb{R}^n$ be a vector function of x. Then the derivative of W along q is given by

$$L_q W \triangleq \frac{\partial W}{\partial x} q(x)$$

and is called the Lie derivative of W w.r.t. q

This notation makes it easy to successively apply derivatives along different vector fields:

$$L_g L_f h = \frac{\partial(L_f h)}{\partial x} g(x)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x} h\right) g$$

$$L_f^2 h = \frac{\partial(L_f h)}{\partial x} f$$

$$L_f^k h = \frac{\partial L_f^{k-1} h}{\partial x} f$$

$$L_f^0 h = h$$

Consider the nonlinear system

$$\dot{x} = f(x) + g(x) u$$
$$y = h(x) \qquad (7.1)$$

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x} f + \frac{\partial h}{\partial x} g u = L_f h + L_g h u$$

if $L_g h = 0$ then

$$\dot{y} = L_f h$$

differentiate again yields

$$\ddot{y} = L_f^2 h + L_g L_f h u$$

Definition:

The nonlinear system (7.1) is said to have relative degree ρ $1 \le \rho \le n$ in a region $D_0 \subset D$

if

$$L_g L_f^{i-1} h = 0 \quad i = 1, 2, \ldots, \rho - 1$$

$$L_g L_f^{\rho-1} h \neq 0 \qquad \text{for all } x \in D_0$$

Example 13.1

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \varepsilon(1-x_1^2)x_2 + u \qquad \varepsilon > 0$$

case 1:

$$y = x_1$$

$$\dot{y} = \dot{x}_1 = x_2$$
$$\ddot{y} = \dot{x}_2 = -x_1 + \varepsilon(1-x_1^2)x_2 + u$$

$\therefore$ relative degree is $\rho = 2$

case 2:

$$y = x_2$$
$$\dot{y} = \dot{x}_2 = -x_1 + \varepsilon(1-x_1^2)x_2 + u$$

$\therefore$ relative degree is $\rho = 1$

case 3:

$$y = x_1 + x_2^2$$
$$\dot{y} = \dot{x}_1 + 2x_2\dot{x}_2 = x_2 + 2x_2\left(-x_1 + \varepsilon(1-x_1^2)x_2 + u\right)$$
$$= x_2 - 2x_1x_2 + 2\varepsilon x_2(1-x_1^2)x_2 + x_2 u$$

$\therefore$ relative degree is $\rho = 1$ on $D_0 = \{x : x_2 \neq 0\}$
relative degree is $\rho = 2$ on $\{x_2 = 0\}$

Example 13.2

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = x_2 + u$$
$$y = x_1$$

$$\dot{y} = \dot{x}_1 = x_1$$

$$\ddot{y} = \dot{x}_1 = x_1$$

$$\Rightarrow y^{(k)} = x_1$$

$\therefore$ no relative degree!

Note that the output is completely decoupled from the input.

Example 13.3

$$\dot{x}_1 = -a x_1 + u$$
$$\dot{x}_2 = -b x_2 + k - c x_1 x_3$$
$$\dot{x}_3 = \theta x_1 x_2$$
$$y = x_3$$

$$\dot{y} = \dot{x}_3 = \theta x_1 x_2$$
$$\ddot{y} = \theta \dot{x}_1 x_2 + \theta x_1 \dot{x}_2$$
$$= -\theta a x_1 x_2 + \theta x_2 u - \theta b x_1 x_2 + \theta x_1 k - \theta c x_1^2 x_3$$

$\therefore$ relative degree is $\rho = 2$ on $\{x_2 \neq 0\}$

If a nonlinear system has relative degree ρ then

$$y^{(\rho)} = L_f^{(\rho)} h + L_g L_f^{(\rho-1)} h \, u$$

where $L_g L_f^{(\rho-1)} h \neq 0$

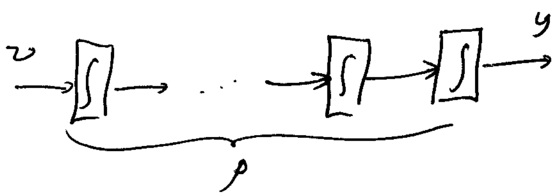
$\therefore$ Letting

$$u = \frac{1}{L_g L_f^{(\rho-1)} h} \left\{ -L_f^{(\rho)} h + v \right\}$$

gives

$$y^{(\rho)} = v$$

which is a chain of ρ integrators:



which can be effectively stabilized with

$$v = -k_{\rho-1} y^{(\rho-1)} - k_{\rho-2} y^{(\rho-2)} - \cdots - k_1 \dot{y} - k_0 y$$

with appropriately chosen k_i

Lets consider the linear system given by the transfer function

$$H(s) = \frac{N(s)}{D(s)}$$

To be concrete let

$$H(s) = \frac{2s+4}{s^3+3s^2+4s+5}$$

By division we have

$$\begin{array}{r|l} & \tfrac{1}{2}s^2 + \tfrac{1}{2}s + 1 \\ 2s+4 & s^3 + 3s^2 + 4s + 5 \\ & \underline{s^3 + 2s^2} \\ & \quad s^2 + 4s + 5 \\ & \quad \underline{s^2 + 2s} \\ & \qquad 2s + 5 \\ & \qquad \underline{2s + 4} \\ & \qquad\quad 1 \end{array}$$

$$\therefore \quad \frac{D}{N} = \frac{s^3+3s^2+4s+5}{2s+4} = \tfrac{1}{2}s^2 + \tfrac{1}{2}s + 1 + \frac{1}{2s+4}$$

$$\triangleq Q + \frac{R}{N}$$

or

$$D = QN + R$$

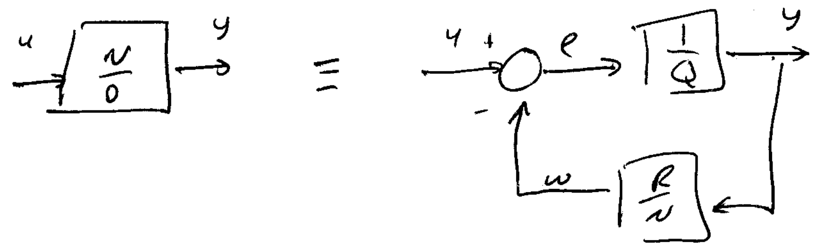
$$\therefore \quad H = \frac{N}{D} = \frac{N}{QN+R} = \frac{1/Q}{1 + \frac{1}{Q}\frac{R}{N}}$$

where

$$\frac{1}{Q} = \frac{1}{\frac{1}{2}s^2 + \frac{1}{2}s + 1} = \frac{2}{s^2+s+2}$$

$$\frac{R}{N} = \frac{1}{2s+4} = \frac{1/2}{s+2}$$

$\therefore$



Note that $\frac{1}{Q} := \frac{2}{s^2+s+2}$ has a state space realization of

$$\dot{\xi} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} \xi + \begin{pmatrix} 0 \\ 1 \end{pmatrix} 2e$$

$$y = (1 \quad 0)\,\xi$$

or

$$\dot{\xi} = \Big[\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A_c} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B_c} \underbrace{(-2 \quad -1)}_{\lambda^T} \Big] \xi + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B_c} \underbrace{2}_{b_m} e$$

$$y = \underbrace{(1 \quad 0)}_{C_c}\, \xi$$

Also $R/N = \frac{1/2}{s+2}$ has state space realization

$$\dot{\eta} = \underbrace{(-2)}_{A_o} \eta + \underbrace{1/2}_{B_o} y$$

$$w = \underbrace{(1)}_{C_o} \eta$$

Combining it all together gives

$$\dot{\xi} = [A_c + B_c \lambda^T]\xi + B_c b_m [u - C_o \eta]$$

$$\dot{\eta} = A_o \eta + B_o C_c \xi$$

$$y = C_c \xi$$

or

$$\dot{\xi} = A_c \xi + B_c (\lambda^T \xi - b_m C_o \eta + b_m u)$$

$$\dot{\eta} = A_o \eta + B_o C_c \xi$$

$$y = C_c \xi$$

Now if $$u = \frac{1}{b_m}\left[-\lambda^T \xi + b_m C_0 \eta \xi + v\right]$$

then $$\dot{\xi} = A_c \xi + B_c v$$

or $$\frac{d}{dt}\begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{y} \\ v \end{pmatrix}$$

$$\Rightarrow \quad \ddot{y} = v$$

ie we have linearized the input-output map.

However the remaining system is

$$\dot{\eta} = A_0 \eta + B_0 C_c \xi$$

where the eigenvalues of A_0 are the roots of $N(s)$, ie the zero's of $H(s)$

The η-subsystem is stable if the zeros of $N(s)$ are in the LHP, i.e., if the original system is "minimum phase."

Our goal is to have a similar decomposition for nonlinear systems.

We will use the change of variables

$$z = T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ \hline h(x) \\ \vdots \\ L_f^{\rho-1} h \end{bmatrix} = \begin{bmatrix} \phi(x) \\ \hline \psi(x) \end{bmatrix} = \begin{bmatrix} \eta \\ \hline \xi \end{bmatrix} \qquad (13.1)$$

where $\phi_i(x)$ are chosen such that

$T(x)$ is a diffeomorphism and

$$\frac{\partial \phi_i}{\partial x} g = 0 \qquad 1 \le i \le n-\rho$$

Thm 13.1 Consider the system $\dot{x} = f + gu$, $y = h(x)$

with relative degree $\rho \le n$ in D.

If $\rho = n$, then for every $x_0 \in D$, a neighborhood N of x_0 exists such that the map

$$T(x) = \begin{bmatrix} h \\ L_f h \\ \vdots \\ L_f^{n-1} h \end{bmatrix}$$

is a diffeomorphism on N.

If $\rho \le n$, then $\exists$ $\phi_1 \cdots \phi_{n-\rho}$ st (13.1) is a ~~diff~~ diffeomorphism on N.

i.e., we are guaranteed to be able to do the change of variables. Unfortunately the proof of the theorem is not constructive!

$$\therefore \quad \dot{\eta}_1 = \frac{\partial \phi_1}{\partial x}^T f$$

$$\vdots$$

$$\dot{\eta}_{n-\rho} = \frac{\partial \phi_{n-\rho}}{\partial x}^T f$$

$$\dot{\xi}_1 = \xi_2$$

$$\dot{\xi}_2 = \xi_3$$

$$\vdots$$

$$\dot{\xi}_\rho = L_f^\rho h + L_g L_f^{\rho-1} h \, u$$

$$y = \xi_1$$

or

$$\left.\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A_c \xi + B_c \gamma(x) \left[u - \alpha(x) \right] \\ y &= C_c \xi \end{aligned}\right] \quad \text{NORMAL FORM}$$

where $\gamma = L_g L_f^{(\rho-1)} h$ and $\alpha = -\dfrac{L_f^\rho h}{L_g L_f^{(\rho-1)} h}$

In normal form, the nonlinear dynamics have an external part (ξ) and an internal part (η).

The external part is linearized by

$$u = \alpha(x) + \frac{1}{\gamma(x)} v$$

For the internal dynamics $\dot{\eta} = f_0(\eta, \xi)$, setting $\xi = 0$ gives

$$\dot{\eta} = f_0(\eta, 0)$$

which are called the "zero dynamics".

If the zero dynamics are asymptotically stable, then the system is said to be minimum phase.

~~Note that~~ The zero dynamics can be characterized in the original state variables.

Note that if

$$y \equiv 0 \quad (\text{ie} \quad y = 0, \ \dot{y} = 0, \ldots \ y^{(\rho)} = 0)$$

$$\Rightarrow \xi(t) \equiv 0$$

$$\Rightarrow 0 \equiv \dot{\xi} = A_c \xi + B_c \gamma(x) [u - \alpha(x)]$$

$$= B_c \gamma(x) [u - \alpha(x)] \equiv 0$$

$$\Rightarrow u \equiv \alpha(x) = - \frac{L_f^{(\rho)} h}{L_g L_f^{(\rho-1)} h}$$

Letting $Z^* = \{ x \in D_0 \mid h(x) = L_f h = \cdots L_f^{(\rho-1)} h = 0 \}$

The zero dynamics are given by

$$\dot{x} = f + g\alpha \Big|_{x \in Z^*}$$

Example 13.5

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u$$
$$y = x_2$$

Note that $\dot{y} = \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u \quad \therefore \rho = 1$

Let $\xi = y = x_2$

and $\eta = x_1$

$\therefore$ the system is already in normal form:

$$\dot{\eta} = \xi$$
$$\dot{\xi} = -\eta + \varepsilon(1 - \eta_1^2)\xi + u$$
$$y = \cancel{\eta}\,\xi$$

as ~~such~~ an input-output linearizing control is

$$u = \eta - \varepsilon(1 - \eta^2)\xi + v$$

resulting in

$$\dot{\eta} = \xi$$
$$\dot{\xi} = v$$
$$y = \xi$$

The zero dynamics are given by $\dot{\eta} = 0 \quad \therefore$

the system is <u>not</u> minimum phase.

Example 13.6

$$\dot{x}_1 = -x_1 + \frac{2+x_3^2}{1+x_3^2}u$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = x_1 x_3 + u$$

$$y = x_2$$

$$= \begin{pmatrix} -x_1 \\ x_3 \\ x_1 x_3 \end{pmatrix} + \begin{pmatrix} \frac{2+x_3^2}{1+x_3^2} \\ 0 \\ 1 \end{pmatrix} u$$

$$\dot{y} = \dot{x}_2 = x_3$$

$$\ddot{y} = \dot{x}_3 = x_1 x_2 + u \qquad \therefore \quad \rho = 2$$

$$L_f h = [0 \;\; 1 \;\; 0] \begin{pmatrix} * \\ x_3 \\ * \end{pmatrix} = x_3$$

$$L_f^2 h = [0 \;\; 0 \;\; 1] \begin{bmatrix} * \\ * \\ x_1 x_3 \end{bmatrix} = x_1 x_3$$

$$L_g L_f h = [0 \;\; 0 \;\; 1] \begin{pmatrix} * \\ 0 \\ 1 \end{pmatrix} = 1$$

$$\therefore \quad \alpha(x) = -\frac{x_1 x_3}{1}$$

and $Z^* = \{ x : x_2 = x_3 = 0 \}$

$\therefore$ the zero dynamics are given by

$$\dot{x}_1 = -x_1 + \frac{2+x_3^2}{1+x_3^2}(-x_1 x_3)\Big|_{x_1 = x_2 = 0}$$

$$\Rightarrow \quad \dot{x}_1 = -x_1$$

$\therefore$ The system is minimum phase.

Full State Linearization

We have seen that the system

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

can be transformed into normal form:

$$\dot{\eta} = f_0(\eta, \xi)$$
$$\dot{\xi} = A_c \xi + B_c \gamma(x)[u - \alpha(x)]$$
$$y = C_c \xi$$

where $y = h(x)$ is an output of interest.

If our objective is full state linearization, then we are not ~~necessarily~~ necessarily interested in any particular output.

Therefore the output equation $h(x)$ becomes a design variable.

Thm The single input nonlinear system $\dot{x} = f(x) + g(x)u$

is full-state linearizable iff there exists a function $h(x)$ s.t. the system

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

has relative degree $\rho = n$

proof: If there exist ~~~~ such an h, then in normal form we have

$$\dot{\xi} = A_c \xi + \gamma(x)[u - \alpha(x)]$$
$$y = C_c \xi$$

which is linearized by $u = \alpha(x) + \frac{1}{\gamma(x)} v$.

If on the other hand, $\dot{x} = f + gu$ is feedback linearizable, then $\exists$ a change of variable $\xi = S(x)$ that transforms the system into

$$\dot{\xi} = A\xi + B\bar{\gamma}(x)[u - \bar{\alpha}(x)]$$

where (A,B) - controllable and $\bar{\gamma}(x) \neq 0$.

Find M that transforms (A,B) into control canonical form, ie $MAM^{-1} = A_c + B_c\lambda^T$, $MB = B^c$

Then $z = M\xi = MS(x) \triangleq T(x)$

gives

$$\begin{aligned}\dot{z} = M\dot{\xi} &= MAM^{-1}z + MB\bar{\gamma}(x)[u - \bar{\alpha}(x)] \\ &= A_c z + B_c\lambda^T z + B_c\bar{\gamma}(x)[u - \bar{\alpha}(x)] \\ &= A_c z + B_c\bar{\gamma}(x)\left[\frac{\lambda^T z}{\bar{\gamma}(x)} + u - \bar{\alpha}(x)\right] \\ &= A_c z + B_c\gamma(x)[u - \alpha(x)]\end{aligned}$$

where $\gamma(x) = \bar{\gamma}(x)$ and $\alpha(x) = \bar{\alpha}(x) - \frac{\lambda^T z}{\bar{\gamma}(x)}$

~~~~

$$z = T(x) \Rightarrow \dot{z} = \frac{\partial T}{\partial x}\dot{x}$$

or

$$A_c z + B_c\gamma(x)[u - \alpha(x)] = \frac{\partial T}{\partial x} f + \frac{\partial T}{\partial x} g u$$
~~~~

Taking $u = 0$ we get

$$A_c T(x) + \cancel{\ldots} - B_c \gamma(x)\alpha(x) = \frac{\partial T}{\partial x} f \qquad (3.1)$$

and $$\frac{\partial T}{\partial x} g = B_c \gamma(x) \qquad (3.2)$$

eq (3.1) becomes

$$\begin{pmatrix} \frac{\partial T_1^T f}{\partial x} \\ \vdots \\ \frac{\partial T_n^T f}{\partial x} \end{pmatrix} = \begin{bmatrix} T_2(x) \\ T_3(x) \\ \vdots \\ T_n(x) \\ -\gamma(x)\alpha(x) \end{bmatrix}$$

eq (3.2) become

$$\begin{pmatrix} \frac{\partial T_1}{\partial x} g \\ \vdots \\ \frac{\partial T_n}{\partial x} g \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \gamma(x) \neq 0 \end{pmatrix} \qquad (3.3)$$

Set $h(x) = T_1(x)$

Then $T_2 = L_f T_1 = L_f h$

$T_3 = L_f^2 h$

$\vdots$

$T_n = L_f^{n-1} h$

$\therefore$ (3.3) gives that

$$L_g L_f^{i-1} h = 0 \qquad i = 1, \ldots, n-1$$

$$L_g L_f^{n-1} h = \gamma(x) \neq 0$$

$\therefore$ The system has relative degree $= n$. $\square$

The proof is also semi-constructive.

If we can find a smooth $h(x)$ s.t.

$$L_g L_f^{i-1} h = 0 \qquad i = 1, \ldots, n-1$$

$$L_g L_f^{n-1} h \neq 0$$

then we have a feedback linearizing transformation

Example.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} a \sin x_2 \\ -x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

we need to find an h such that

$$L_g h = 0$$
$$L_g L_f h \neq 0$$

ie $\frac{\partial h}{\partial x}^T g = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\partial h}{\partial x_2} = 0$ (ie $h(x_1, x_2) = h(x_1)$)

and ~~$L_g L_f h$~~ $\frac{\partial (L_f h)}{\partial x} g = \begin{bmatrix} \frac{\partial}{\partial x_1}(L_f h) & \frac{\partial}{\partial x_2}(L_f h) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$= \frac{\partial (L_f h)}{\partial x_2}$$

$$= \frac{\partial}{\partial x_2}\left(\frac{\partial h}{\partial x_1}(a \sin x_2) + \frac{\partial h}{\partial x_2}(-x_1^2) \right)$$

since $h(x_1, x_2) = h(x_1)$ we have $\frac{\partial h}{\partial x_2} = 0$

$$\therefore \quad = \frac{\partial}{\partial x_2}\left(\frac{\partial h}{\partial x_1} a \sin x_2 \right)$$

$$= \frac{\partial h}{\partial x_1} a \cos x_2 \neq 0$$

So pick any $h(x_1)$ s.t. $\frac{\partial h}{\partial x_1} \neq 0$

and it will be valid when ~~a~~ $\cos x_2 \neq 0$

For example $h(x_1) = x_1$ or $h(x_1) = x_1 + x_1^3$

Picking $h(x_1) = x_1$ the linearizing transformation becomes

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = T(x) = \begin{pmatrix} h(x_1) \\ L_f h \end{pmatrix} = \begin{bmatrix} x_1 \\ a \sin x_2 \end{bmatrix}$$

Example — single link manipulator with flexible joints

$$\dot{x} = f + gu = \begin{pmatrix} x_2 \\ -a\sin x_1 - b(x_1 - x_3) \\ x_4 \\ c(x_1 - x_3) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ d \end{pmatrix} u$$

we need to find an $h(x)$ s.t.

$$L_g L_f^{i-1} h = 0 \qquad i = 0, 1, 2$$

$$L_g L_f^3 h \neq 0$$

$$\cancel{L_f h = \frac{\partial h}{\partial x_1} x_2 + \frac{\partial h}{\partial x_2}\left(-a\sin x_1 - b(x_1 - x_3)\right) + \frac{\partial h}{\partial x_3} x_4 + \frac{\partial h}{\partial x_4} c(x_1 - x_3)}$$

$$\Rightarrow \cancel{L_g L_f h =}$$

$$\therefore \quad L_g h = \frac{\partial h}{\partial x_4} d = 0 \quad \Rightarrow \quad \frac{\partial h}{\partial x_4} = 0, \text{ ie } h \text{ is independent of } x_4$$

$$L_f h = \frac{\partial h}{\partial x_1} x_2 + \frac{\partial h}{\partial x_2}\left(-a\sin x_1 - b(x_1 - x_3)\right) + \frac{\partial h}{\partial x_3} x_4$$

$$L_g L_f h = \frac{\partial (L_f h)}{\partial x_4} d = \frac{\partial h}{\partial x_3} d = 0 \Rightarrow h \text{ is independent of } x_3$$

$$L_f^2 h = \frac{\partial (L_f h)}{\partial x_1} x_2 + \frac{\partial (L_f h)}{\partial x_2}\left(-a\sin x_1 - b(x_1 - x_3)\right) + \frac{\partial (L_f h)}{\partial x_3} x_4$$

$$L_g L_f^2 h = \frac{\partial (L_f h)}{\partial x_3} d = +b \frac{\partial h}{\partial x_2} d = 0 \Rightarrow h \text{ is independent of } x_2$$

$$L_f^3 h = \frac{\partial (L_f^2 h)}{\partial x_1} x_2 + \frac{\partial L_f^2 h}{\partial x_2}\left(-a\sin x_1 - b(x_1 - x_3)\right) + \frac{\partial L_f^2 h}{\partial x_3} x_4$$

$$L_g L_f^3 h = \frac{\partial (L_f^2 h)}{\partial x_3} = + b \frac{\partial (L_f h)}{\partial x_2} = b \frac{\partial h}{\partial x_1} \neq 0 \text{ if } \frac{\partial h}{\partial x_1} \neq 0$$

Therefore ~~and~~ any $h(x) = h(x_1)$ s.t. $\frac{\partial h}{\partial x_1} \neq 0$ will work.

Let $h(x_1) = x_1$

then
$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = T(x) = \begin{pmatrix} h(x_1) \\ L_f h \\ L_f^2 h \\ L_f^3 h \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ \frac{\partial h}{\partial x_1} f_1 = 1 \cdot x_2 = x_2 \\ \frac{\partial x_2}{\partial x_1} f_1 + \frac{\partial x_2}{\partial x_2} f_2 + \frac{\partial x_2}{\partial x_3} f_3 + \frac{\partial x_2}{\partial x_4} f_4 = -a \sin x_1 - b(x_1 - x_3) \\ [-a \cos x_1 \; - b] x_2 + b x_4 \end{pmatrix}$$

So
$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ -a \sin x_1 - b(x_1 - x_3) \\ [a \cos x_1 - b] x_2 + b x_4 \end{pmatrix}$$

Suppose that the system

$$\dot{x} = f + g u$$ is feedback linearizable,

ie $\exists$ $z = T(x)$ s.t.

$$\dot{z} = A_c z + B\gamma(x)\left[u - \alpha(x)\right]$$

The system is stabilized by choosing

$$u = \alpha(x) + \frac{1}{\gamma(x)}(-kz) \triangleq \alpha(x) + \beta(x)(-kz) \qquad (7.1)$$

to get $\dot{z} = (A - Bk)z$

However implementation of (7.1) requires exact knowledge of the dynamics.

Suppose instead that we implement

$$u = \hat{\alpha}(x) - \hat{\beta}(x) K \hat{T}(z)$$

where $\hat{\alpha}$, $\hat{\beta}$, $\hat{T}$ are approximations of α, β, T

Then

$$\dot{z} = Az + B\gamma(x)\left[\hat{\alpha} - \hat{\beta}K\hat{T} + \underline{\alpha} - \underline{\beta KT} - \alpha + \beta KT - \underline{\alpha}\right]$$

$$= (A - Bk)z + B\gamma(x)\left[(\hat{\alpha} - \alpha) + \beta KT - \hat{\beta}K\hat{T}\right]$$

$$= (A - Bk)z + B\delta(z)$$

where $$\delta(z) \triangleq \gamma(x)\left[(\hat{\alpha} - \alpha) + (\beta K T - \hat{\beta} K \hat{T})\right]$$

Lemma 13.3

If k is s.t. $A-Bk$ is Hurwitz and $P = P^T > 0$ is the solution to

$$P(A-Bk) + (A-Bk)^T P = -I$$

1) If $\|\delta(z)\| \le k\|z\|$ where $0 < k < \frac{1}{2\|PB\|_2}$

then ~~...~~ the perturbed system is globally exponentially stable

2) If $\|\delta(z)\| \le k\|z\| + \varepsilon$ for $0 < k < \frac{1}{2\|PB\|}$

then the system is globally ultimately bounded

proof:

Let $V = z^T P z$ then

$$\dot{V} = \dot{z}^T P z + z^T P \dot{z}$$

$$= z^T\left[(A-Bk)^T P + P(A-Bk)\right] z + 2 z^T PB\,\delta(z)$$

$$\le -\|z\|^2 + 2\|PB\|\,\|z\|\,\|\delta(z)\|$$

1) If $\|\delta(z)\| < \frac{1}{2\|PB\|}\|z\|$, then

$$\dot{V} = -\left(1 - \frac{k}{2\|PB\|}\right)\|z\|^2 < 0$$

2) If $\|\delta(z)\| \le k\|z\| + \varepsilon$

then

$$\dot{V} \le -\|z\|^2 + 2k\|PB\|\,\|z\|^2 + 2\varepsilon\|PB\|\,\|z\|$$
$$+ \theta\|z\|^2 - \theta\|z\|^2$$

$$= -(1-\theta)\|z\|^2 + \left[2k\|PB\|\,\|z\|^2 + 2\varepsilon\|PB\|\,\|z\| - \theta\|z\|^2\right)$$

$\dot{V} < 0$ if

$$\left[2k\|PB\| - \theta\right]\|z\|^2 + 2\varepsilon\|PB\|\,\|z\| < 0$$

if $0 < \theta < 1$ s.t.

$\theta - 2k\|PB\| > 0$ ie pick θ s.t.

$$2k\|PB\| < \theta < 1$$

which is possible since $k < \frac{1}{2\|PB\|}$

then $\Rightarrow$ $(\theta - 2k\|PB\|)\,\|z\| > 2\varepsilon\|PB\|$

or $$\|z\| > \frac{2\varepsilon\|PB\|}{\theta - 2k\|PB\|}$$

$\therefore$ The system is globally ~~ul~~ ultimately bounded.

Tracking.

Suppose that the nonlinear system

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

has been transformed into normal form:

$$\dot{\eta} = f_0(\eta, \xi)$$
$$\dot{\xi} = A_c \xi + B_c \frac{1}{\beta(x)} [u - \alpha(x)]$$
$$y = C_c \xi$$

Assume (1) $f_0(0,0) = 0$

(2) reference signal y_r and its derivatives (up to ρ) are bounded

Let $Y_r = \begin{pmatrix} y_r \\ \dot{y}_r \\ \vdots \\ y_r^{(\rho-1)} \end{pmatrix}$ and let $e = \xi - Y_r$

Then $\dot{e} = \dot{\xi} - \begin{pmatrix} \dot{y}_r \\ \ddot{y}_r \\ \vdots \\ y_r^{(\rho)} \end{pmatrix} = \dot{\xi} - A_c Y_c +$

$$= A_c e + B_c \left\{ \frac{1}{\beta(x)} [u - \alpha(x)] - y_r^{(\rho)} \right\}$$

$\therefore$ Let $u = \alpha(x) + \beta(x) \left[-Ke + y_r^{(\rho)} \right]$

to get $\dot{e} = (A_c - B_c K) e$

The ~~chosen~~ closed loop system is
given by

$$\dot{\eta} = f_0(\eta, e + Y_c)$$
$$\dot{e} = (A_c - B_c K)e$$

If we assume that $\dot{\eta} = f_0(\eta, \xi)$ is
input-to-state stable, then
asymptotic tracking is ensured.