2nd Order Systems

A useful visualization tool is the "phase plot" of a 2nd order system.

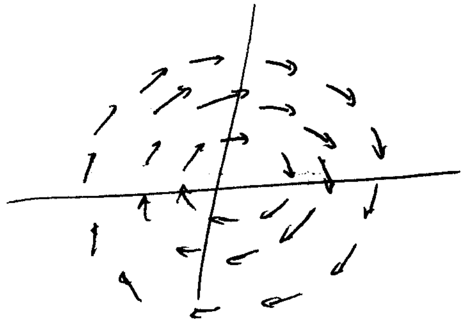
A phase plot is a plot of all possible trajectories of a system.

Consider the 2nd order system

$$\dot{x} = f(x) \qquad \text{where } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \; f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

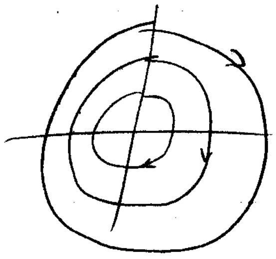
If at every x, we plot $f(x)$ ~~at~~

eg



and then draw tangent ~~the~~ curves to these vectors

eg



We have a qualitative plot of all solutions to the equation $\dot{x} = f(x)$ for different initial conditions

Phase plots can be generated in Matlab calling ode 45 with a sample of initial conditions.

Qualitative behavior of Linear Systems

Consider the linear system

$$\dot{x} = Ax \quad , \quad x(0) = x_0$$

The solution to this ode is

$$\dot{x} = e^{At} x_0$$

Using eigenvalue decomposition, A can be written as

$$A = M^{-1} J M$$

where Jr is the real Jordan form

$$\therefore \quad \dot{x} = M e^{J_r t} M^{-1} x_0$$

or, letting

$$z = M^{-1} x \quad \text{(similarity transformation)}$$

$$\dot{z} = e^{J_r t} z_0$$

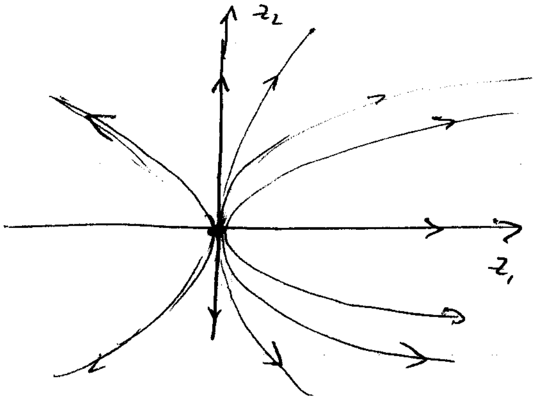
Jr can have the following forms:

$$\underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{\text{real, non repeated eigenvalues}} \qquad \underbrace{\begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix}}_{\text{real, repeated eigenvalues}} \qquad \underbrace{\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}}_{\text{complex eigenvalues}}$$

Case 1 eigenvalues real, non-repeated

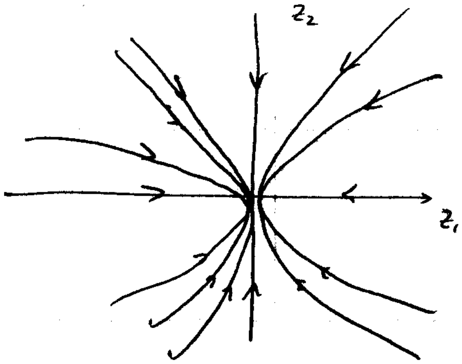
$$z_1(t) = e^{\lambda_1 t} z_{10} \qquad \left(\begin{array}{l} \dot z_1 = \lambda_1 z_1 \\ \dot z_2 = \lambda_2 z_2 \end{array} \right)$$
$$z_2(t) = e^{\lambda_2 t} z_{20}$$

Therefore the phase portraits look ~~the~~ like



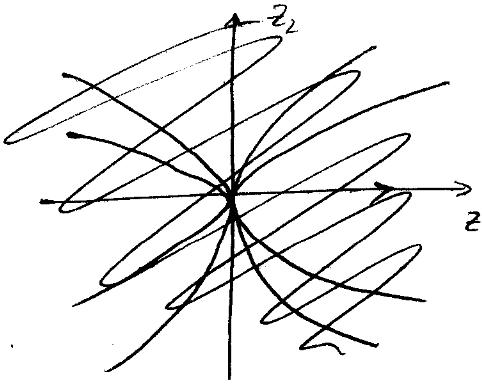
$\lambda_1 > \lambda_2 > 0$

unstable node



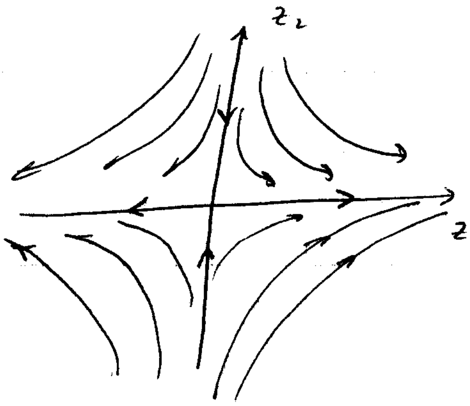
$0 > \lambda_2 > \lambda_1$

Stable node



~~$\lambda_1 > 0$, $\lambda_2 < 0$~~

~~$|\lambda_1| > |\lambda_2|$~~



$\lambda_1 > 0$, $\lambda_2 < 0$

$|\lambda_1| > |\lambda_2|$

Saddle Node

In the original x_1, x_2 coordinates the z_1, z_2 axes are stretched to align with the eigenvectors.

$$-\beta \;]$$
$$\alpha \;]$$

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corresponds to the case of
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(1.31)

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. The slope of the curve

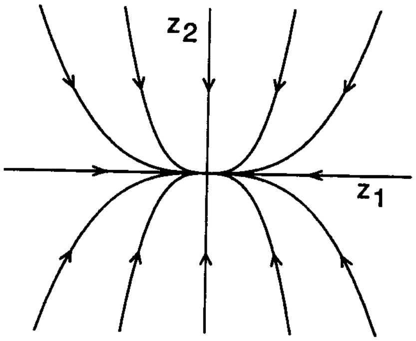


Figure 1.14: Phase portrait of a stable node in modal coordinates.

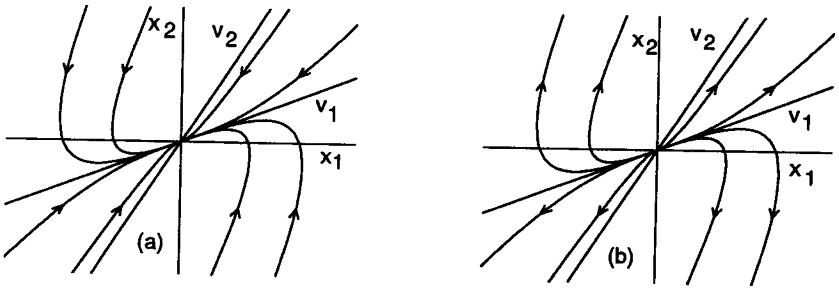


Figure 1.15: Phase portraits for (a) a stable node; (b) an unstable node.

Since $[(\lambda_2/\lambda_1) - 1]$ is positive, the slope of the curve approaches zero as $|z_1| \to 0$ and approaches ∞ as $|z_1| \to \infty$. Therefore, as the trajectory approaches the origin, it becomes tangent to the z_1-axis; as it approaches ∞, it becomes parallel to the z_2-axis. These observations allow us to sketch the typical family of trajectories shown in Figure 1.14. When transformed back into the x-coordinates, the family of trajectories will have the typical portrait shown in Figure 1.15(a). Note that in the x_1–x_2 plane, the trajectories become tangent to the slow eigenvector v_1 as they approach the origin and parallel to the fast eigenvector v_2 far from the origin. In this case, the equilibrium point $x = 0$ is called a *stable node.*

When λ_1 and λ_2 are positive, the phase portrait will retain the character of Figure 1.15(a) but with the trajectory directions reversed, since the exponential terms $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ grow exponentially as t increases. Figure 1.15(b) shows the phase portrait for the case $\lambda_2 > \lambda_1 > 0$. The equilibrium point $x = 0$ is referred to in this

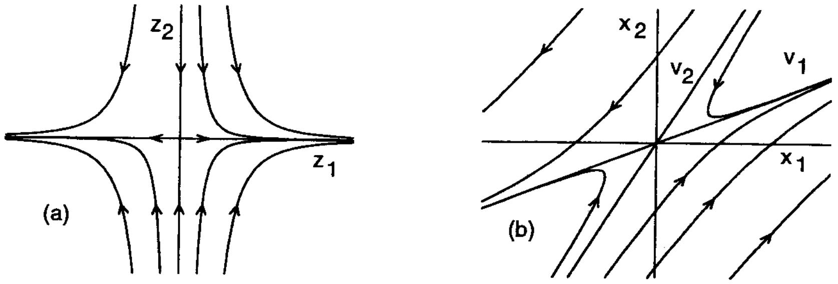


Figure 1.16: Phase portrait of a saddle point (a) in modal coordinates; (b) in original coordinates.

case as an *unstable node.*

Suppose now that the eigenvalues have opposite signs. In particular, let $\lambda_2 < 0 < \lambda_1$. In this case, $e^{\lambda_1 t} \to \infty$ while $e^{\lambda_2 t} \to 0$ as $t \to \infty$. Hence, we call λ_2 the stable eigenvalue and λ_1 the unstable eigenvalue. Correspondingly, v_2 and v_1 are called the stable and unstable eigenvectors, respectively. The trajectory equation (1.31) will have a negative exponent (λ_2/λ_1). Thus, the family of trajectories in the z_1–z_2 plane will take the typical form shown in Figure 1.16(a). Trajectories have hyperbolic shapes. They become tangent to the z_1-axis as $|z_1| \to \infty$ and tangent to the z_2-axis as $|z_1| \to 0$. The only exception to these hyperbolic shapes are the four trajectories along the axes. The two trajectories along the z_2-axis are called the stable trajectories since they approach the origin as $t \to \infty$, while the two trajectories along the z_1-axis are called the unstable trajectories since they approach infinity as $t \to \infty$. The phase portrait in the x_1–x_2 plane is shown in Figure 1.16(b). Here the stable trajectories are along the stable eigenvector v_2 and the unstable trajectories are along the unstable eigenvector v_1. In this case, the equilibrium point is called a *saddle*.

Case 2. Complex eigenvalues: $\lambda_{1,2} = \alpha \pm j\beta$.

The change of coordinates $z = M^{-1}x$ transforms the system (1.30) into the form

$$\begin{aligned} \dot{z}_1 &= \alpha z_1 - \beta z_2 \\ \dot{z}_2 &= \beta z_1 + \alpha z_2 \end{aligned}$$

The solution of this equation is oscillatory and can be expressed more conveniently

Case 2 Complex eigenvalues

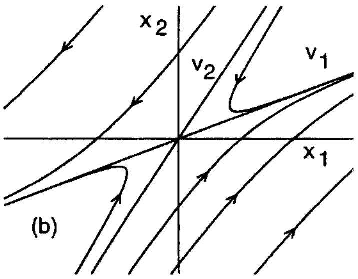
$\lambda_{1,2} = \alpha \pm j\beta$

In z-coordinates we get

$$\left.\begin{array}{l} \dot{z}_1 = \alpha z_1 - \beta z_2 \\ \dot{z}_2 = \alpha z_1 + \beta z_2 \end{array}\right\} \text{ circular phase portrait}$$

In polar coordinates we have

$$\left.\begin{array}{l} \dot{r} = \alpha r \\ \dot{\theta} = \beta \end{array}\right\| \Rightarrow \begin{array}{l} r(t) = r_0 e^{\alpha t} \\ \theta(t) = \theta_0 + \beta t \end{array}$$



) in modal coordinates; (b) in original

posite signs. In particular, let $\lambda_2 <$
0 as $t \to \infty$. Hence, we call λ_2 the
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ctor v_2 and the unstable trajectories
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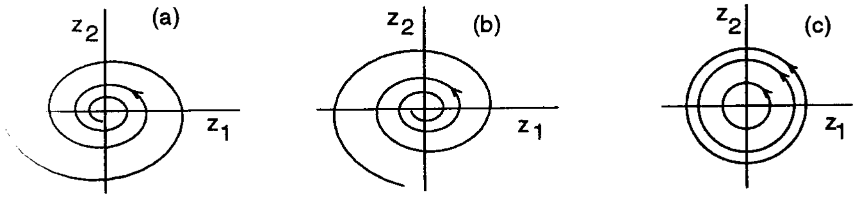
$\pm j\beta$.

ns the system (1.30) into the form

βz_2
αz_2

can be expressed more conveniently

~~20~~ (7)



ure 1.17: Typical trajectories in the case of complex eigenvalues.
0; (b) $\alpha > 0$; (c) $\alpha = 0$.

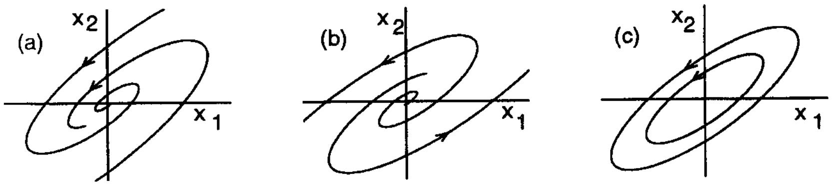


Figure 1.18: Phase portraits for (a) a stable focus; (b) an unstable focus; (c) a center.

In the polar coordinates

$$r = \sqrt{z_1^2 + z_2^2}, \quad \theta = \tan^{-1}\left(\frac{z_2}{z_1}\right)$$

we have two uncoupled first-order differential equations:

$$\begin{aligned} \dot{r} &= \alpha r \\ \dot{\theta} &= \beta \end{aligned}$$

The solution for a given initial state (r_0, θ_0) is given by

$$r(t) = r_0 e^{\alpha t}, \qquad \theta(t) = \theta_0 + \beta t$$

which defines a logarithmic spiral in the z_1–z_2 plane. Depending on the value of α, the trajectory will take one of the three forms shown in Figure 1.17. When $\alpha < 0$, the spiral converges to the origin; when $\alpha > 0$, it diverges away from the origin. When $\alpha = 0$, the trajectory is a circle of radius r_0. Figure 1.18 shows the trajectories in the x_1–x_2 plane. The equilibrium point $x = 0$ is referred to as a *stable focus* if $\alpha < 0$, *unstable focus* if $\alpha > 0$, and *center* if $\alpha = 0$.

Case 3 repeated non zero eigenvalues

$$\dot{z}_1 = \lambda z_1 + k z_2$$

$$\dot{z}_2 = \lambda z_2$$

$$\therefore \quad z_2(t) = e^{\lambda t} z_{20}$$

$$z_1(t) = e^{\lambda t} z_{10} + k t e^{\lambda t} z_{20}$$

Repeated non-zero eigenvalues

~~22~~ 9

2 eigenvectors

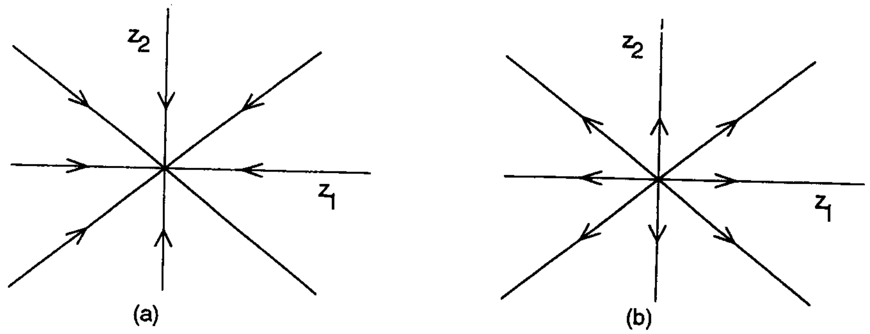


Figure 1.19: Phase portraits for the case of nonzero multiple eigenvalues when $k = 0$: (a) $\lambda < 0$; (b) $\lambda > 0$.

Case 3. Nonzero multiple eigenvalues: $\lambda_1 = \lambda_2 = \lambda \neq 0$.

The change of coordinates $z = M^{-1}x$ transforms the system (1.30) into the form

$$\begin{aligned}\dot{z}_1 &= \lambda z_1 + k z_2 \\ \dot{z}_2 &= \lambda z_2\end{aligned}$$

whose solution, for a given initial state (z_{10}, z_{20}), is given by

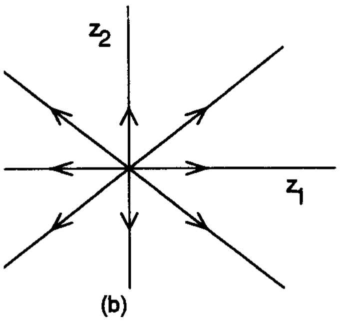
$$z_1(t) = e^{\lambda t}(z_{10} + k z_{20} t), \qquad z_2(t) = e^{\lambda t} z_{20}$$

Eliminating t, we obtain the trajectory equation

$$z_1 = z_2 \left[\frac{z_{10}}{z_{20}} + \frac{k}{\lambda} \ln \left(\frac{z_2}{z_{20}} \right) \right]$$

Figure 1.19 shows the form of the trajectories when $k = 0$, while Figure 1.20 shows their form when $k = 1$. The phase portrait has some similarity with the portrait of a node. Therefore, the equilibrium point $x = 0$ is usually referred to as a stable node if $\lambda < 0$ and unstable node if $\lambda > 0$. Note, however, that the phase portraits of Figures 1.19 and 1.20 do not have the asymptotic slow-fast behavior that we saw in Figures 1.14 and 1.15.

Before we discuss the degenerate case when one or both of the eigenvalues are zero, let us summarize our findings about the qualitative behavior of the system when the equilibrium point $x = 0$ is isolated. We have seen that the system can display six qualitatively different phase portraits, which are associated with different types of equilibria: stable node, unstable node, saddle point, stable focus, unstable focus, and center. The type of equilibrium point is completely specified by the location of the eigenvalues of A. Note that the global (throughout the phase plane)



multiple eigenvalues when $k = 0$:

$\lambda_1 = \lambda_2 = \lambda \neq 0$.

the system (1.30) into the form

$_2$

is given by

$_2(t) = e^{\lambda t} z_{20}$

$$\left.\frac{z_2}{z_{20}}\right)\Bigg]$$

en $k = 0$, while Figure 1.20 shows
some similarity with the portrait
0 is usually referred to as a stable
however, that the phase portraits
tic slow-fast behavior that we saw

one or both of the eigenvalues are
qualitative behavior of the system
have seen that the system can dis-
which are associated with different
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is completely specified by the lo-
bal (throughout the phase plane)

Repeated non zero eigenvalues
one - eigenvector

~~23~~ 10

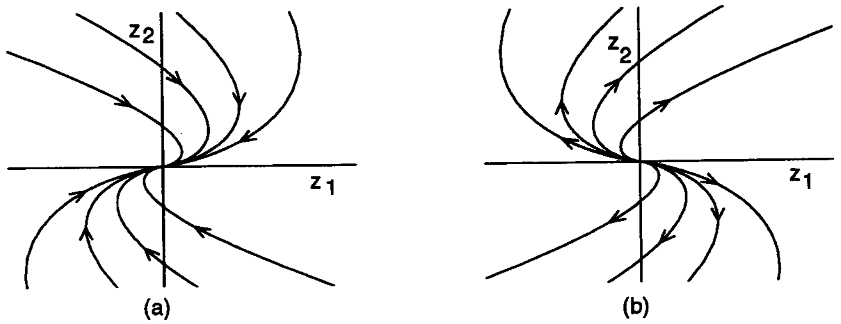


Figure 1.20: Phase portraits for the case of nonzero multiple eigenvalues when $k = 1$: (a) $\lambda < 0$; (b) $\lambda > 0$.

qualitative behavior of the system is determined by the type of equilibrium point. This is a characteristic of linear systems. When we study the qualitative behavior of nonlinear systems in the next section, we shall see that the type of equilibrium point can only determine the qualitative behavior of the trajectories in the vicinity of that point.

Case 4. One or both eigenvalues are zero.

When one or both eigenvalues of A are zero, the phase portrait is in some sense degenerate. In this case, the matrix A has a nontrivial null space. Any vector in the null space of A is an equilibrium point for the system; that is, the system has an equilibrium subspace rather than an equilibrium point. The dimension of the null space could be one or two; if it is two, the matrix A will be the zero matrix. This is a trivial case where every point in the plane is an equilibrium point. When the dimension of the null space is one, the shape of the Jordan form of A will depend on the multiplicity of the zero eigenvalue. When $\lambda_1 = 0$ and $\lambda_2 \neq 0$,the matrix M is given by $M = [v_1, v_2]$ where v_1 and v_2 are the associated eigenvectors. Note that v_1 spans the null space of A. The change of variables $z = M^{-1}x$ results in

$$\begin{aligned} \dot{z}_1 &= 0 \\ \dot{z}_2 &= \lambda_2 z_2 \end{aligned}$$

whose solution is

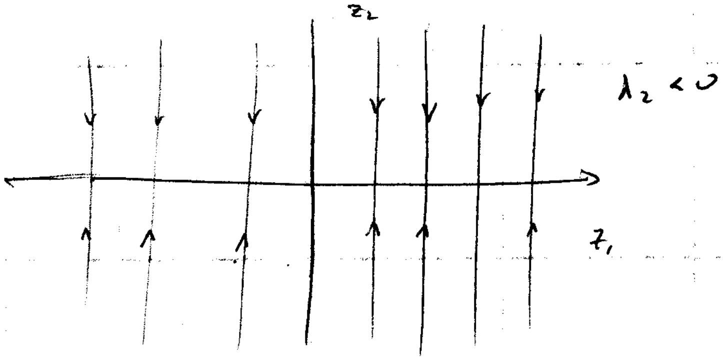
$$z_1(t) = z_{10}, \quad z_2(t) = z_{20}e^{\lambda_2 t}$$

The exponential term will grow or decay, depending on the sign of λ_2. Figure 1.21 shows the phase portrait in the x_1–x_2 plane. All trajectories converge to the equilibrium subspace when $\lambda_2 < 0$, and diverge away from it when $\lambda_2 > 0$.

Case 4   one or both eigenvalues are zero

(one eigenvalue is zero)

$$\dot z_1 = 0$$
$$\dot z_2 = \lambda_2 z_2$$



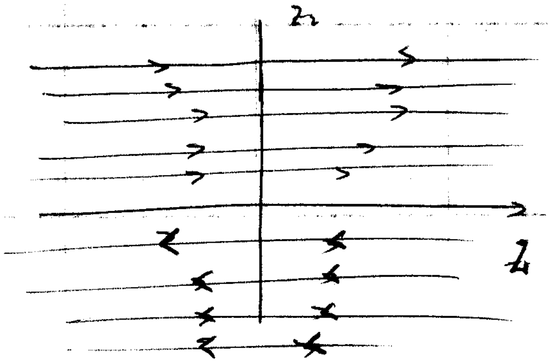
stable subspace (Nontrivial null space of A)

two eigenvalues are zero, ~~as~~

eg double integrator $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

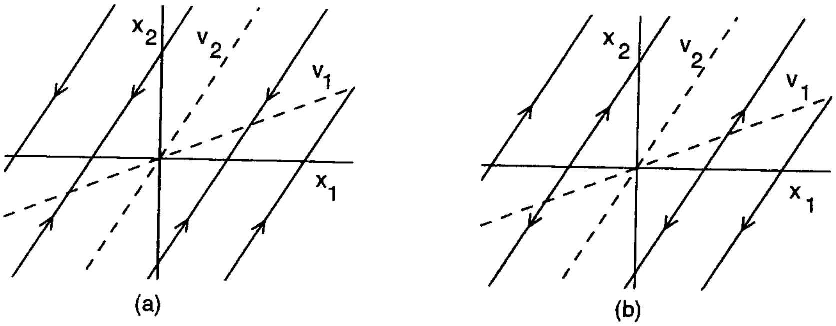
in this case

$$\dot z_1 = z_2$$
$$\dot z_2 = 0$$



Note that arrows are same direction on each line.

12 ~~25~~



(a) (b)

Figure 1.21: Phase portraits for (a) $\lambda_1 = 0,\ \lambda_2 < 0$; (b) $\lambda_1 = 0,\ \lambda_2 > 0$.

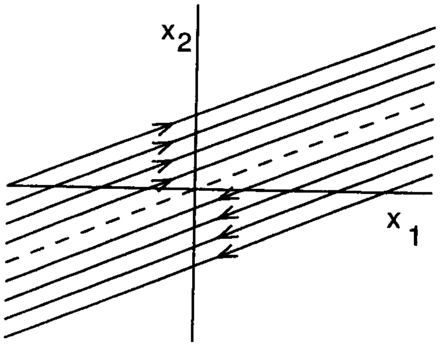


Figure 1.22: Phase portrait when $\lambda_1 = \lambda_2 = 0$.

When both eigenvalues are at the origin, the change of variables $z = M^{-1}x$ results in

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= 0 \end{aligned}$$

whose solution is

$$z_1(t) = z_{10} + z_{20}t, \quad z_2(t) = z_{20}$$

The term $z_{20}t$ will increase or decrease, depending on the sign of z_{20}. The z_1-axis is the equilibrium subspace. Figure 1.22 shows the phase portrait in the x_1–x_2 plane; the dashed line is the equilibrium subspace. The phase portrait in Figure 1.22 is quite different from that in Figure 1.21. Trajectories starting off the equilibrium subspace move parallel to it.

Small perturbations of A

If A does not perfectly model the system, ie the real system is given by

$$\dot{x} = (A+\Delta A)x$$

where $\| \Delta A \|$ is small, how does the equilibrium Structure change?

Result: If A does not have zero eigenvalues then small perturbations do not affect the ~~eq~~ equilibrium structure + the system is called "structurally stable".

If A has ~~a~~ zero eigenvalues then small perturbations may result in, positive, negative or zero eigenvalues. ~~& then~~
~~Therefore~~ therefore the system is not structurally stable.

Def: The origin $x=0$ is a hyperbolic equilibrium point ~~if~~ $\dot{x} = Ax$ if A has no ~~zero~~ eigenvalues. with zero real part, ~~other~~ otherwise it is non hyperbolic.

What does all of this have to do with nonlinear systems?

It turns out that equilibria of nonlinear systems can be characterized by linearization if the linearization is hyperbolic (proved later in course)

Let $\dot{x} = f(x)$ have an equilibrium at $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$

Using Taylor series expansion about p we get

$$\dot{x}_1 = f_1(p) + \frac{\partial f_1}{\partial x_1}(p)(x_1 - p_1) + \frac{\partial f_1}{\partial x_2}(p)(x_2 - p_2) + HOT$$

$$\dot{x}_2 = f_2(p) + \frac{\partial f_2}{\partial x_1}(p)(x_1 - p_1) + \frac{\partial f_2}{\partial x_2}(p)(x_2 - p_2) + HOT$$

Since p is an equilibrium $f(p) = 0$

$$\therefore \quad \dot{x} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \frac{\partial f_1}{\partial x_2}(p) \\ \frac{\partial f_2}{\partial x_1}(p) & \frac{\partial f_2}{\partial x_2}(p) \end{bmatrix}}_{\triangleq A}(x - p) + HOT$$

Let $y = x - p \Rightarrow \dot{y} = \dot{x}$

then $\dot{y} = Ay + \underbrace{HOT}_{\text{possible perturbations}}$

in fact we could factor HOT as

$\Delta A(y)\, y$

to get $\dot{y} = (A + \Delta A(y))\, y$

When y is small $\|\Delta A(y)\|$ is small.

Therefore if $Re\{eig(A)\} \neq 0$ then the qualitative structure of the equilibrium can be determined by the eigenvalues of A.

Example Inverted pendulum

$$f(x) = \begin{pmatrix} x_2 \\ -g/l \sin x_1 - \frac{k}{ml} x_2 \end{pmatrix} \; ; \; p = \begin{pmatrix} n\pi \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos p_1 & -\frac{k}{ml} \end{pmatrix}$$

Let $g/l = \frac{k}{ml} = 1$

if $p = \begin{pmatrix} n\pi \\ 0 \end{pmatrix}$ n-even then $A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

eigenvalues are $-.5 \pm j0.866$

therefore $p = \begin{pmatrix} n\pi \\ 0 \end{pmatrix}$ n-even are stable nodes

if $p = \begin{pmatrix} n\pi \\ 0 \end{pmatrix}$ n-odd then $A = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

eigenvalues are -1.618 , 0.618

therefore $p = \begin{pmatrix} n\pi \\ 0 \end{pmatrix}$ n-odd are unstable nodes.

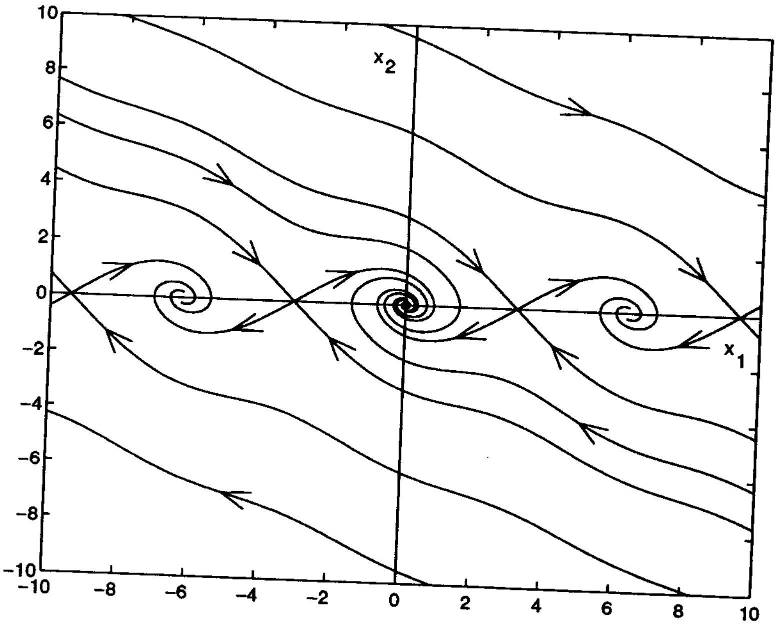


Figure 1.26: Phase portrait of the pendulum equation of Example 1.3.

and forth between the vertical boundaries of this strip. For example, consider the trajectory starting from point *a* in Figure 1.27. Follow this trajectory until it hits the right boundary at point *b*; then identify point *c* on the left boundary, which has the same value of the x_2 coordinate as point *b*. We then continue following the trajectory from *c*. This process is equivalent to following the trajectory on a cylinder formed by cutting out the phase portrait in Figure 1.27 along the vertical lines $x_1 = \pm\pi$ and pasting them together. When represented in this manner, the pendulum equation is said to be represented in a cylindrical phase space rather than a phase plane. From the phase portrait we see that, except for special trajectories which terminate at the "unstable" equilibrium position $(\pi, 0)$, all other trajectories tend to the "stable" equilibrium position $(0, 0)$. Once again, the "unstable" equilibrium position $(\pi, 0)$ cannot be maintained in practice, because noise would cause the trajectory to diverge away from that position. △

1.2.3 Qualitative Behavior Near Equilibrium Points

Examination of the phase portraits in Examples 1.2 and 1.3 shows that the qualitative behavior in the vicinity of each equilibrium point looks just like those we saw in Section 1.2.1 for linear systems. In particular, in Figure 1.24 the trajectories near

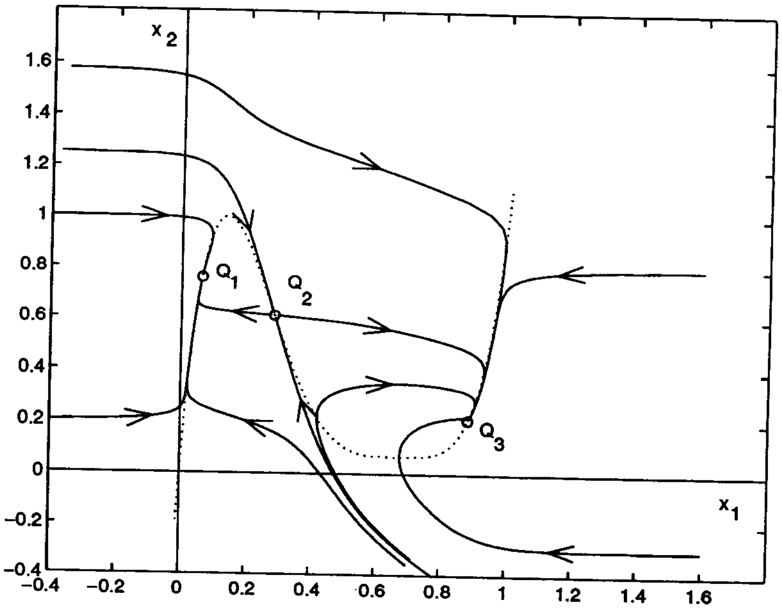


Figure 1.24: Phase portrait of the tunnel diode circuit of Example 1.2.

Setting $\dot{x}_1 = \dot{x}_2 = 0$ and solving for the equilibrium points, it can be verified that there are three equilibrium points at $(0.063, 0.758)$, $(0.285, 0.61)$, and $(0.884, 0.21)$. The phase portrait of the system, generated by a computer program, is shown in Figure 1.24. The three equilibrium points are denoted in the portrait by Q_1, Q_2, and Q_3, respectively. Examination of the phase portrait shows that, except for two special trajectories which tend to the point Q_2, all trajectories eventually tend to either the point Q_1 or the point Q_3. The two special trajectories converging to Q_2 form a curve that divides the plane into two halves. All trajectories originating from the left side of the curve will tend to the point Q_1, while all trajectories originating from the right side will tend to point Q_3. This special curve is called a *separatrix* because it partitions the plane into two regions of different qualitative behavior.[11] In an experimental setup, we shall observe one of the two steady-state operating points Q_1 or Q_3, depending on the initial capacitor voltage and inductor current. The equilibrium point at Q_2 is never observed in practice because the ever-present physical noise would cause the trajectory to diverge from Q_2 even if it were possible to set up the exact initial conditions corresponding to Q_2. This tunnel

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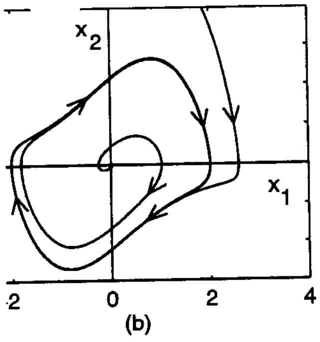
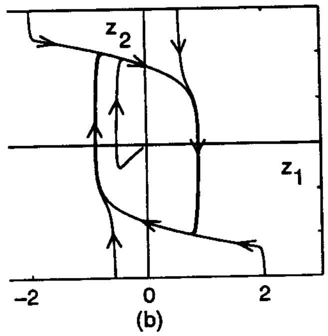
[11] In general, the state plane decomposes into a number of regions, within each of which the trajectories may show a different type of behavior. The curves separating these regions are called separatrices.



llator: (a) $\epsilon = 0.2$; (b) $\epsilon = 1.0$.



lla   with $\epsilon = 5.0$: (a) in x_1–x_2

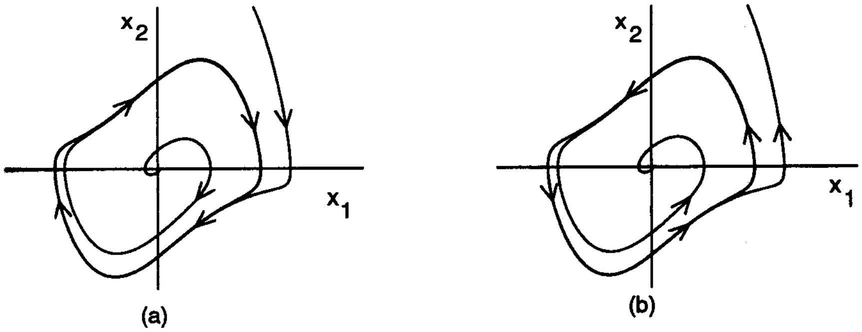
) show the phase portraits of the

(1.32)

$^2_1)x_2$ (1.33)

all value of 0.2, a medium value
s, the phase portraits show that
ctories starting off the orbit. For
lose to a circle of radius 2. This
ım value of $\epsilon = 1.0$, the circular
gure 1.30(b). For the large value
shown in Figure 1.31(a). A more

Limit Cycles ⑱ ~~㊲~~



(a) (b)

Figure 1.32: (a) A stable limit cycle; (b) an unstable limit cycle.

revealing phase portrait in this case can be obtained when the state variables are chosen as $z_1 = i_L$ and $z_2 = v_C$, resulting in the state equation

$$\begin{aligned} \dot{z}_1 &= \frac{1}{\epsilon} z_2 \\ \dot{z}_2 &= -\epsilon(z_1 - z_2 + \tfrac{1}{3} z_2^3) \end{aligned}$$

The phase portrait in the z_1–z_2 plane for $\epsilon = 5.0$ is shown in Figure 1.31(b). The closed orbit is very close to the curve $z_1 = z_2 - \frac{1}{3} z_2^3$ except at the corners, where it becomes nearly vertical. This vertical portion of the closed orbit can be viewed as if the closed orbit jumps from one branch of the curve to the other as it reaches the corner. Oscillations where this *jump phenomenon* takes place are usually referred to as *relaxation oscillations.* This phase portrait is typical for large values of ϵ (say, $\epsilon > 3.0$). △

The closed orbit we have seen in Example 1.7 is different from what we have seen in the harmonic oscillator. In the case of the harmonic oscillator, there is a continuum of closed orbits, while in the Van der Pol example there is only one isolated periodic orbit. An isolated periodic orbit is called a *limit cycle.*[15] The limit cycle of the Van der Pol oscillator has the property that all trajectories in the vicinity of the limit cycle ultimately tend toward the limit cycle as $t \to \infty$. A limit cycle with this property is classically known as a stable limit cycle. We shall also encounter *unstable limit cycles*, which have the property that all trajectories starting from points arbitrarily close to the limit cycle will tend away from it as $t \to \infty$; see Figure 1.32. To see an example of an unstable limit cycle, consider the Van der Pol equation in reverse time; that is,

$$\dot{x}_1 = -x_2$$

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[15] A formal definition of limit cycles will be given in Chapter 6.