2.2 Existence and Uniqueness

Consider the ode

$$\dot{x} = x^{1/3} \qquad x(0) = 0$$

Claim:
$x(t) = 0$ is a solution
proof: obvious

Claim: $x(t) = \left(\frac{2t}{3}\right)^{3/2}$ is a solution

proof
$$\dot{x} = 3/2 \left(2t/3\right)^{1/2} \cdot \frac{2}{3} = \left(\frac{2t}{3}\right)^{1/2}$$
$$x^{1/3} = \left(\left(\frac{2t}{3}\right)^{3/2}\right)^{1/3} = \left(\frac{2t}{3}\right)^{1/2}$$

Therefore the solution is not unique!
We need a ~~the~~ condition on $f(t,x)$ to guarantee existence and uniqueness of solutions

~~The Ba~~

Lipschitz Conditions

$f(x)$ is said to be globally Lipschitz if
$$\|f(x)-f(y)\| \le L\|x-y\| \qquad \forall x,y \in \mathbb{R}^n$$
and where L is a constant.

$f(x)$ is locally Lipschitz on an open and connected set $D \subset \mathbb{R}^n$ if at each $x \in D$, $\exists$ a neighborhood $D_0 \subset D$ s.t. $\exists L_0$ s.t.
$$\|f(x)-f(y)\| \le L_0\|x-y\| \qquad \forall y \in D_0$$

f is ~~Lipschitz~~ Lipschitz on $W \subset \mathbb{R}^n$ if

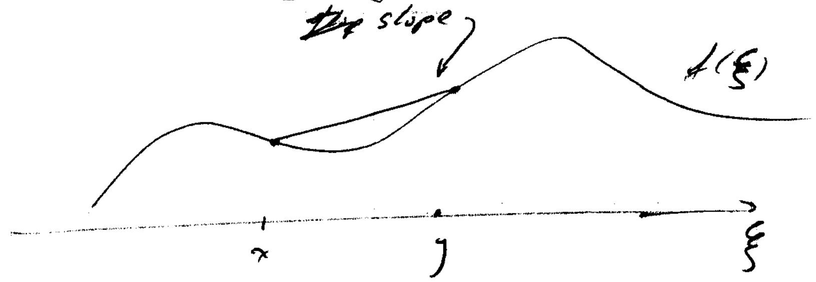
$$\|f(x) - f(y)\| \leq L \|x - y\| \quad \forall x, y \in W$$

ie the <u>same</u> Lipschitz constant works for all points in W.

$f(t, x)$ is (Locally) Lipschitz on $[a, b] \times D \subset \mathbb{R} \times \mathbb{R}^n$ if ~~it~~ for each x, the same L_0 works $\forall t \in [a, b]$

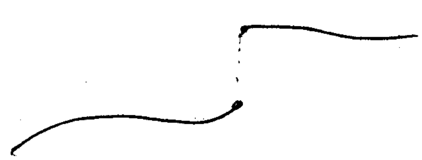
<u>Meaning</u>: ~~If~~ $f: \mathbb{R} \to \mathbb{R}$ then

$$\underbrace{\frac{|f(y) - f(x)|}{|x - y|}}_{\text{slope}} \leq L$$



The Lipschitz condition essentially means that the slope of f in x is bounded.

Therefore, discontinuous functions are not Lipschitz at the point of discontinuity.



On the other hand, if $|f'(x)|$ is uniformly bounded, then f is globally Lipschitz.

Thm 3.1 ~~Thm~~ Local Existence and Uniqueness

Let $f(t,x)$ be piecewise cont. in t and satisfy

$$\|f(t,x) - f(t,y)\| \le L \|x-y\|$$

$$\forall\, x,y \in B = \{x \in \mathbb{R}^n : \|x - x_0\| \le r\} \quad \forall\, t \in [t_0,t_1].$$

Then, there exists some $\delta > 0$ such that

$$\dot{x} = f(t,x) \quad \text{with} \quad x(t_0) = x_0$$

has a unique solution over $[t_0, t_0+\delta]$.

proof: By integration we have

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

$$\triangleq (Px)(t)$$

We need to show that P is a contraction map.

First we need a Banach space.

The set of continuous functions $\mathcal{X} = C[a,b]$ with norm $\|x\|_{\mathcal{X}} = \max_{t\in[a,b]} \|x(t)\|$ is a Banach Space.

We need a closed subset of $\mathcal{X}$. Define

$$S = \{x \in \mathcal{X} : \|x - x_0\|_{\mathcal{X}} \le r\}$$

(Note $x \in S \Rightarrow$ $x(t) \in B \quad \forall\, t \in [t_0,t_1]$

$\Rightarrow \|x(t) - x_0\| \le r$

Lemma ~~2.2~~ 3.1 Let $f : [a,b] \times D \to \mathbb{R}^m$ be continuous on $D \subset \mathbb{R}^n$. Suppose $\frac{\partial f}{\partial x}$ exists and is continuous on $[a,b] \times D$. If for a convex set $W \subset D$, $\exists\, L \geq 0$ s.t.

$$\left\| \frac{\partial f}{\partial x} \right\| \leq L$$

on $[a,b] \times W$ then

$$\| f(t,x) - f(t,y) \| \leq L \|x - y\|$$

$\forall\, t \in [a,b]$, $x, y \in W$.

Lemma ~~2.3~~ 3.2 Let $f(t,x)$ be cont. on $[a,b] \times D$. If $\frac{\partial f}{\partial x}$ is cont. on $[a,b] \times D$, then f is locally Lipschitz in x on $[a,b] \times D$.

Lemma ~~2.4~~ 3.3 Let $f(t,x)$ be cont on $[a,b] \times \mathbb{R}^n$. If $\frac{\partial f}{\partial x}$ is cont on $[a,b] \times \mathbb{R}^n$, then f is globally Lipschitz iff $\frac{\partial f}{\partial x}$ is uniformly bounded on $[a,b] \times \mathbb{R}^n$.

Thm 2.3 Global Existence + Uniqueness

Suppose $f(t,x)$ is piecewise cont. in t an globally Lipschitz, ie

$$\|f(t,x) - f(t,y)\| \le L\|x-y\|$$

$$\|f(t,x_0)\| \le h$$

$\forall\, x,y \in \mathbb{R}^n$, $\forall\, t \in [t_0, t_1]$, then $\dot{x} = f(t,x)$

$x(t_0) = x_0$ has a unique solution over $[t_0, t_1]$.

proof: Since f is globally Lipschitz, we can make r ~~as large~~ in the proof of Thm 2.2, arbitrarily large. Therefore

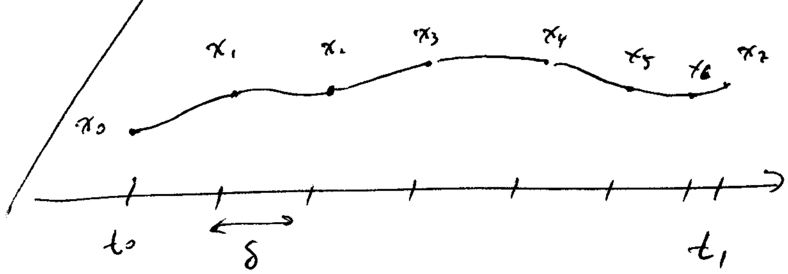
$$\delta \le \min\left\{ \frac{\rho}{L},\ \frac{r}{Lr+h},\ t_1 - t_0 \right\} = \min\left\{ \frac{\rho}{L},\ t_1 - t_0 \right\}$$

(Note $\displaystyle\lim_{r\to\infty} \frac{r}{Lr+h} = \lim_{r\to\infty} \frac{1}{L+h/r} = \frac{1}{L} > \frac{\rho}{L}$

since $\rho < 1$.)

If $t_1 - t_0 < \rho/L$, choose $\delta = t_1 - t_0$.

Otherwise subdivide $t_1 - t_0$ into segments of length $\delta \le \rho/L$ apply Thm 2.2 repeatedly

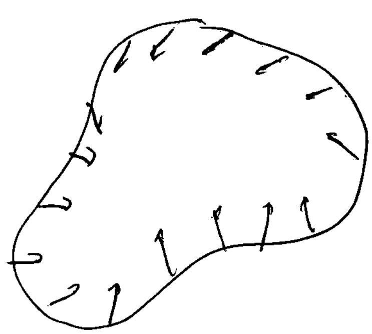


Since the global Lipschitz condition is difficult to satisfy, we would like extra conditions that guarantee global existence + uniqueness when f is locally Lipschitz.

<u>Thm ~~2.4~~ 3.3</u> Let $f(t,x)$ be piecewise cont in t and locally Lipschitz in x $\forall\, t \geq t_0$ and at $x \in D \subset \mathbb{R}^n$. Let $W \subset D$ be compact, and $x_0 \in W$, and suppose that every solution of $\dot{x} = f(t,x)$ $x(t_0) = x_0$ lies in W. Then there is a unique solution ~~that stays in~~ $\forall\, t \geq t_0$

The difficulty is verifying that all solutions remain in W. However, Lyapunov theory will enable us to guarantee this, without ~~sol.~~ ~~finding an~~ solving the ode.

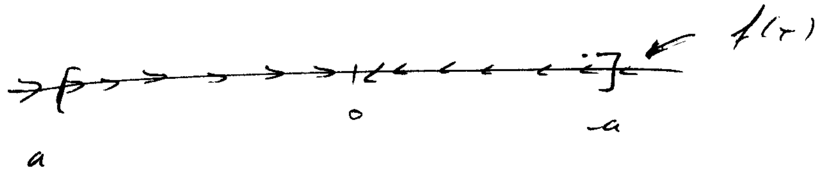
<u>Basic idea:</u> Look at the direction of f at the boundary of W:



If f ~~does~~ points into the interior of W $\forall\, x \in \partial W$, then the solution will never leave W.

<u>Example</u>

$\dot{x} = -x^3 \qquad x_0 \in [-a, a]$



$f(x) = -x^3$ is locally Lipschitz ~~on [-a,a]~~ ~~[-a,a] and~~ + it is clear that solutions starting in $[-a, a]$ will never leave $[-a,a]$, thus a unique solution exists.

Gronwall-Bellman Inequality (Appendix A)

Motivation: Consider the nonlinear differential equation

$$\dot{x} = f(x) \quad , \quad x(0) = x_0$$

then $$x(t) = x_0 + \int_0^t f(x(\tau))\, d\tau$$

Notice that $x(t)$ appears on both sides of equation. The Gronwall-Bellman inequality removes $x(t)$ from the RHS.

Lemma 2.1 Let $\lambda : [a,b] \to \mathbb{R}$ be continuous and $\mu : [a,b] \to \mathbb{R}$ be continuous and nonnegative. If $y : [a,b] \to \mathbb{R}$ is cont. and satisfies

$$y(t) \leq \lambda(t) + \int_a^t \mu(s)\, y(s)\, ds$$

then

$$y(t) \leq \lambda(t) + \int_a^t \lambda(s)\mu(s) \left[e^{\int_s^t \mu(\tau)\, d\tau} \right] ds$$

If $\lambda(t) \equiv \lambda$ is a constant then

$$y(t) \leq \lambda\, e^{\int_a^t \mu(\tau)\, d\tau}$$

If $\mu(t) \equiv \mu > 0$ is a constant, then

$$y(t) \leq \lambda\, e^{\mu(t-a)}$$

Proof: Let's first recall how to solve linear odes via integrating factors.

Suppose we are given

$$\dot{z} = az + bu \qquad , a, b - \text{constants}$$

then

$$\dot{z} - az = bu$$

Multiplying both sides by the integrating factor

$$e^{-a(t-\sigma)} \quad \text{gives}$$

$$e^{-a(t-\sigma)}(\dot{z} - az) = e^{-a(t-\sigma)} bu$$

or $\frac{d}{dt}\left\{ e^{-a(t-\sigma)} z \right\} = e^{-a(t-\sigma)} bu(t)$

Integrating from α to t gives

$$\int_\alpha^t \left[\frac{d}{ds} e^{-a(s-\sigma)} z(s)\right] ds = \int_\alpha^t e^{-a(s-\sigma)} bu(s)\, ds$$

$$\Rightarrow \quad e^{-a(t-\sigma)} z(t) - e^{-a(\alpha-\sigma)} z(\alpha) = \int_\alpha^t e^{-a(s-\sigma)} b\, u(s)\, ds$$

$$\Rightarrow \quad z(t) = e^{+a(t-\sigma)} e^{-a(\alpha-\sigma)} z(\alpha) + e^{a(t-\sigma)} \int_\alpha^t e^{-a(s-\sigma)} bu(s) ds$$

$$\Rightarrow \quad z(t) = e^{(t-\alpha)} z(\alpha) + \int_\alpha^t e^{a(t-s)} bu(s)\, ds$$

Now suppose that ~~the~~ $a = a(t)$, i.e.

$$\dot{z} = a(t)\, z + b(t)\, u$$

then

$$\dot{z} - a(t) z(t) = b(t) u(t)$$

Using the integrating factor $e^{-\int_\sigma^t a(\tau)d\tau}$ gives

$$e^{-\int_\sigma^t a(\tau)d\tau}\left[\dot{z} - a(t) z(t)\right] = \cancel{b(t)u(t)}\; e^{-\int_\sigma^t a(\tau)d\tau}\, b(t)\, u(t)$$

or

$$\frac{d}{dt}\left\{ e^{-\int_\sigma^t a(\tau)d\tau} z(t)\right\} = e^{-\int_\sigma^t a(\tau)d\tau}\, b(t)\, u(t)$$

Integrating from α to t gives

$$e^{-\int_\sigma^t a(\tau)d\tau} z(t) - e^{-\int_\sigma^\alpha a(\tau)d\tau} z(\alpha) = \int_\alpha^t e^{-\int_\sigma^s a(\tau)d\tau}\, b(s) u(s)\, ds$$

$$\Rightarrow\; z(t) = e^{\int_\alpha^t a(\tau)d\tau} z(\alpha) + \int_\alpha^t e^{\int_s^t a(\tau)d\tau}\, b(s) u(s)\, ds$$

Returning to

$$y(t) \le \lambda(t) + \underbrace{\int_a^t \mu(s)\, y(s)\, ds}_{\triangleq\, z(t)}$$

Let $\quad v(t) = \lambda(t) + z(t) - y(t) \ge 0$

then $\quad \dot{z} = \mu(t)\, y(t) = \mu(t)\, z(t) + [\mu(t)\lambda(t) - \mu(t)\, v(t)]$

$$\Rightarrow \quad z(t) = e^{\int_a^t \mu(\tau) d\tau} z(a) + \int_a^t e^{\int_s^t \mu(\tau) d\tau} [\mu(s)\lambda(s) - \mu(s)\, v(s)]\, ds$$

but $\quad z(a) = \int_a^a \mu(s)\, y(s)\, ds = 0$

So

$$z(t) = \int_a^t e^{\int_s^t \mu(\tau) d\tau} \mu(s)\lambda(s)\, ds - \int_a^t \underbrace{\underbrace{e^{\int_s^t \mu(\tau) d\tau}}_{\ge 0} \underbrace{\mu(s)}_{\ge 0} \underbrace{v(s)}_{\ge 0}}_{\ge 0} ds$$

$$\therefore \quad z(t) \le \int_a^t e^{\int_s^t \mu(\tau) d\tau} \mu(s)\lambda(s)\, ds$$

$$\Rightarrow \quad y(t) \le \lambda(t) + \int_a^t e^{\int_s^t \mu(\tau) d\tau} \mu(s)\lambda(s)\, ds$$

which is what we wanted to prove

when $\lambda(t) = \lambda$ then

$$y(t) \le \lambda \left(1 + \int_a^t e^{\int_s^t \mu(\tau) d\tau} \mu(s)\, ds\right)$$

$$= \lambda \left(1 - \int_a^t \frac{d}{ds} e^{\int_s^t \mu(\tau) d\tau} ds\right)$$

$$= \lambda \left(1 - e^{\int_t^t \mu(\tau) d\tau} + e^{\int_a^t \mu(\tau) d\tau}\right)$$

$$= \lambda\, e^{\int_a^t \mu(\tau) d\tau}$$

QED.

Continuous dependence on Initial Conditions and Parameters

~~Chap 3~~

Suppose that $f(t,x)$ depends on a number of parameters $\lambda_0 \in \mathbb{R}^p$, eg mass, spring constant, capacitance etc, and suppose that λ_0 is ~~only~~ imprecisely known, ie $\|\lambda - \lambda_0\| < \delta$, if we solve $\dot{x} = f(t, x, \lambda_0)$ will the solution be close to $\dot{x} = f(t, x, \lambda)$?

We would also like to know if near-by initial conditions lead to nearby solutions.

Theorem ~~2.5~~ 3.4 Let $f(t,x)$ be piecewise cont in t and Lipschitz in x on $[t_0, t_1] \times W$ with a Lipschitz constant L, where $W \subset \mathbb{R}^n$ is open an connected. Let $y(t)$ & $z(t)$ be solutions of

$$\dot{y} = f(t,y) \qquad y(t_0) = y_0$$

$$\dot{z} = f(t,z) + g(t,z), \qquad z(t_0) = z_0$$

Such that $y(t), z(t) \in W$ $\forall t \in [t_0, t_1]$.

Suppose $\|g(t,x)\| \leq \mu$ and $\|y_0 - z_0\| \leq \gamma$

Then $\forall t \in [t_0, t_1]$

$$\|y(t) - z(t)\| \leq \gamma e^{L(t-t_0)} + \frac{\mu}{L}\left\{ e^{L(t-t_0)} - 1 \right\}$$

proof:

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds$$

$$z(t) = z_0 + \int_{t_0}^{t} [\, f(s, z(s)) + g(s, z(s)) \,]\, ds$$

$$\Rightarrow \quad \| y(t) - z(t) \| \le \| y_0 - z_0 \| + \int_{t_0}^{t} \| f(s, y(s)) - f(s, z(s)) \| \, ds$$

$$+ \int_{t_0}^{t} \| g(s, z(s)) \| \, ds$$

$$\le \underbrace{\gamma + \mu (t - t_0)}_{\lambda(t)} + \int_{t_0}^{t} \underbrace{L}_{\mu(s)} \| y(s) - z(s) \| \, ds .$$

Now use Gronwall-Bellman inequality

$$\| y(t) - z(t) \| \le \gamma + \mu (t - t_0) + \int_{t_0}^{t} [\gamma + \mu (s - t_0)] L \, e^{\int_s^t L \, d\tau} ds$$

$$= \gamma + \mu (t - t_0) + \int_{t_0}^{t} [\gamma + \mu (s - t_0)] L \, e^{L(t-s)} ds$$

Integration by parts: $\int u\,dv = uv - \int v\,du$

Let $\quad u = -[\gamma + \mu (s - t_0)] \quad \Rightarrow \quad du = -\mu$

$\quad dv = -L\, e^{L(t-s)} ds \quad \Rightarrow \quad v = e^{L(t-s)}$

$$\therefore \quad \| y(t) - z(t) \| \le \gamma + \mu (t - t_0) - [\gamma + \mu (s - t_0)] \, e^{L(t-s)} \Big|_{s=t_0}^{s=t}$$

$$+ \int_{t_0}^{t} \mu \, e^{L(t-s)} ds$$

$$= \gamma + \mu (t - t_0) - [\gamma + \mu (t - t_0)] + \gamma e^{L(t-t_0)}$$

$$+ \frac{\mu}{L} \left(e^{L(t-t_0)} - 1 \right)$$

$$= \gamma e^{L(t-t_0)} + \frac{\mu}{L} \left(e^{L(t-t_0)} - 1 \right)$$

This theorem tells us how fast things can get away from us. Divergence is bounded by an exponential which grows according to the Lipschitz constant.

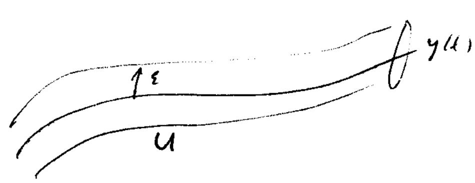
Thm 3.5 Let $f(t,x,\lambda)$ be continuous in (t,x,λ) and locally Lipschitz in x on $[t_0,t_1]\times D \times \{\|\lambda-\lambda_0\| \le c\}$. Suppose $y(t,\lambda_0)$ is a solution to $\dot{x} = f(t,x,\lambda_0)$, $y(t_0,\lambda_0) = y_0 \in D$ and suppose that $y(t,\lambda_0)\in D$ $\forall t \in [t_0,t_1]$. Then $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

if $\|z_0 - y_0\| < \delta$ and $\|\lambda - \lambda_0\| < \delta$

then there is a unique solution $z(t,\lambda)$ of $\dot{x} = f(t,x,\lambda)$, $z(t_0,\lambda) = z_0$

s.t. $\|z(t,\lambda) - y(t,\lambda_0)\| < \varepsilon$ $\forall t \in [t_0,t_1]$.

proof: Define a tube around $y(t,\lambda_0)$



$$U = \{(t,x) \subset [t_0,t_1]\times \mathbb{R}^n : \|x - y(t,\lambda_0)\| \le \varepsilon\}$$

pick ε small enough that $U \subset [t_0,t_1]\times D$.

U is compact, $\Rightarrow$ it is Lipschitz on U with Lipschitz constant L.

f is cont in λ, therefore $\forall \alpha > 0$, $\exists \beta$ s.t.

$\|\lambda - \lambda_0\| < \beta$ $(t,x)\in U$ $\Rightarrow$ $\|f(t,x,\lambda) - f(t,x,\lambda_0)\| \le \alpha$

Let $\|z_0 - y_0\| < \alpha$, then from Thm 2.5

$$\|z(t,\lambda) - y(t,\lambda_0)\| \le \alpha e^{L(t-t_0)} + \frac{\alpha}{L}\{e^{L(t-t_0)} - 1\}$$

$$< \alpha\left(1 + \frac{1}{L}\right) e^{L(t-t_0)} < \varepsilon$$

$$\Rightarrow \quad \alpha < \varepsilon\left(\frac{L}{L+1}\right) e^{-L(t-t_0)}$$

Let $\alpha \le \varepsilon\left(\frac{L}{L+1}\right) e^{-L(t_1-t_0)}$ and $\delta \le \min\{\alpha, \beta\}$

and we are done.

Sensitivity Equations

Let $f(t,x,\lambda)$ be cont. diff in x, λ

then

$$\dot{x} = f(t,x,\lambda) \qquad x(t_0) = x_0$$

$$\Leftrightarrow \quad x(t,\lambda) = x_0 + \int_{t_0}^{t} f(s, x(s,\lambda), \lambda)\, ds$$

$$\Rightarrow \quad \frac{\partial x}{\partial \lambda}(t,\lambda) = \int_{t_0}^{t} \left[\frac{\partial f}{\partial x}(s, x(s,\lambda), \lambda) \frac{\partial x}{\partial \lambda}(t,\lambda) + \frac{\partial f}{\partial \lambda}(s, x(s,\lambda), \lambda) \right] ds$$

Let $A(t,\lambda) = \frac{\partial f}{\partial x}(t,x,\lambda) \qquad B(t,\lambda) = \frac{\partial f}{\partial \lambda}(t,x,\lambda)$

$$x_\lambda = \frac{\partial x}{\partial \lambda}(t,x)$$

then $x_\lambda(t,\lambda) = \int_{t_0}^{t} [A(s,\lambda)\, x_\lambda(s,\lambda) + B(s,\lambda)]\, ds$

$$\Rightarrow \quad \frac{\partial x_\lambda}{\partial t}(t,\lambda) = A(t,\lambda)\, x_\lambda(t,\lambda) + B(t,\lambda)$$

$$x_\lambda(t_0,\lambda) = 0$$

Evaluating at the nominal parameters λ_0

and defining $S(t) = x_\lambda(t,\lambda_0)$

gives

$$\boxed{\dot{S}(t) = A(t,\lambda_0) S(t) + B(t,\lambda_0) \qquad S(t_0) = 0}$$

This is called the sensitivity equation

$S(t)$ are the sensitivity of the solution

with respect to the parameters.

Example 2.8

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -c \sin x_1 - (a + b\cos x_1)\, x_2$$

Find the sensitivity of x_1, x_2 w.r.t a, b, c around the nominal values $a_0 = 1$, $b_0 = 0$, $c_0 = 1$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -c\cos x_1 + b(\sin x_1)\, x_2 & -(a + b\cos x_1) \end{bmatrix}$$

let $\lambda = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\frac{\partial f}{\partial \lambda} = \begin{bmatrix} 0 & 0 & 0 \\ -x_2 & -\cancel{\cos}\, x_2 \cos x_1 & -\sin x_1 \end{bmatrix}$$

The sensitivity equation is

$$\underbrace{\dot{S}}_{2\times 3} = \underbrace{A}_{2\times 2}\,\underbrace{S}_{2\times 3} + \underbrace{B}_{2\times 3}$$

Let $S = \begin{bmatrix} x_3 & x_5 & x_7 \\ x_4 & x_6 & x_8 \end{bmatrix}$

then
$$\begin{bmatrix} \dot{x}_3 & \dot{x}_5 & \dot{x}_7 \\ \dot{x}_4 & \dot{x}_6 & \dot{x}_8 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c_0\cos x_1 + b_0 x_2 \sin x_1 & -(a_0 + b_0 \cos x_1) \end{bmatrix} \begin{bmatrix} x_3 & x_5 & x_7 \\ x_4 & x_6 & x_8 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 \\ -x_2 & -x_2 \cos x_1 & -\sin x_1 \end{bmatrix}$$

gives

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -c \sin x_1 \quad - (a_0 + b_0 \cos x_1)\, x_2$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = (-c_0 \cos x_1 + b_0 x_2 \sin x_1)\, x_3 \quad - (a_0 + b_0 \cos x_1)\, x_4 \qquad - x_2$$

$$\dot{x}_5 = x_6$$

$$\dot{x}_6 = (-c_0 \cos x_1 + b_0 x_2 \sin x_1)\, x_5 \quad - (a_0 + b_0 \cos x_1)\, x_6 \qquad - x_2 \cos x_1$$

$$\dot{x}_7 = x_8$$

$$\dot{x}_8 = (-c_0 \cos x_1 - b_0 x_2 \sin x_1)\, x_7 \quad - (a_0 + b_0 \cos x_1)\, x_8 \qquad - \sin x_1$$

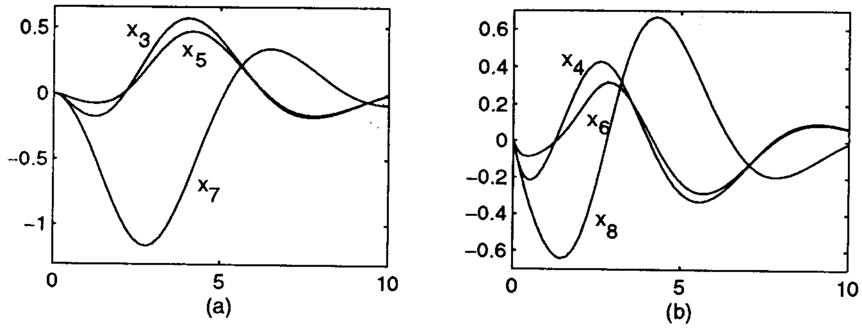


Figure 2.2: Sensitivity functions for Example 2.8.

a scalar differentiable function $v(t)$ satisfies inequality of the form $\dot{v}(t) \leq f(t, v(t))$ for all t in a certain time interval. Such inequality is called a *differential inequality* and a function $v(t)$ satisfying the inequality is called a solution of the differential inequality. The comparison lemma compares the solution of the differential inequality $\dot{v}(t) \leq f(t, v(t))$ to the solution of the differential equation $\dot{u} = f(t, u)$. The lemma applies even when $v(t)$ is not differentiable, but has an upper right-hand derivative $D^+v(t)$ which satisfies a differential inequality. The upper right-hand derivative $D^+v(t)$ is defined in Appendix A.1. For our purposes, it is enough to know two facts:

- if $v(t)$ is differentiable at t, then $D^+v(t) = \dot{v}(t)$;
- if

$$\frac{1}{h}\left|v(t+h) - v(t)\right| \leq g(t, h), \quad \forall\ h \in (0, b]$$

  and

$$\lim_{h \to 0^+} g(t, h) = g_0(t)$$

  then $D^+v(t) \leq g_0(t)$.

The limit $h \to 0^+$ means that h approaches zero from above.

Lemma 2.5 (Comparison Lemma) *Consider the scalar differential equation*

$$\dot{u} = f(t, u), \quad u(t_0) = u_0$$

where $f(t,u)$ is continuous in t and locally Lipschitz in u, for all $t \geq 0$ and all $u \in J \subset R$. Let $[t_0, T)$ (T could be infinity) be the maximal interval of existence