Input - Output Stability

$$\dot{x} = f(t, x, u)$$
$$y = h(t, x, u)$$

How does input-output stability relate to Lyapunov stability

input-output stability — response to input

Lyapunov stability — response to initial conditions

For a linear system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

we have

$$y = \underbrace{Ce^{At}x(0)}_{\text{initial conditions}} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)}_{\text{response to input}}$$

Note that if A - Hurwitz then

$e^{At} \to 0$ so response to initial conditions dies out

Also $\left\| \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau \right\| \le \sup_{0 \le \tau \le t} \|u(\tau)\| \int_0^t \|Ce^{At}B\|\,dt$

∴ if A is Hurwitz then

$$\lim_{t\to\infty} \int_0^t \| C e^{At} B \| dt \to \gamma$$

and

$$\| y(t) \| \leq (\gamma + \|D\|) \sup_{0 \leq \tau \leq t} \| u(\tau) \| + \| C e^{At} x_0 \|$$

$$\Rightarrow \sup_{0 \leq \tau \leq t} \| y(\tau) \| \leq (\gamma + \|D\|) \sup_{0 \leq \tau \leq t} \| u(\tau) \| + \| C e^{At} x_0 \|$$

Let $\| y \|_{\mathcal{L}_\infty} = \sup_{t \geq 0} \| y(t) \|$

Then .

$$\| y \|_{\mathcal{L}_\infty} \leq (\gamma + \|D\|) \| u \|_{\mathcal{L}_\infty} + \| C e^{At} x_0 \|$$

∴ Bounded inputs give bounded outputs

Define

$$\mathcal{L}_p = \left\{ u(t) : \| u \|_{\mathcal{L}_p} = \left(\int_0^\infty \| u(t) \|^p dt \right)^{1/p} < \infty \right\}$$

$$p \in [0, \infty]$$

Note that if $u(t) \to \infty$ as $t \to \infty$ then

$u \notin \mathcal{L}_p$ for any p.

Therefore if we restrict ourselves to signals in some $\mathcal{L}_p$, we will not be able to include unstable systems.

Let $u_\tau(t) = \begin{cases} u(t), & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$

Define the extended $\mathcal{L}_p$ space, denoted $\mathcal{L}_{p,e}$ as

$$\mathcal{L}_{p,e} = \{ u : u_\tau \in \mathcal{L}_p, \forall \tau \ge 0 \}$$

Example $u(t) = t \notin \mathcal{L}_\infty$, However $u_\tau(t) \in \mathcal{L}_\infty$. for any finite τ $\Rightarrow$ $u(t) = t \in \mathcal{L}_{\infty,e}$.

Def. A mapping $H : \mathcal{L}_e \to \mathcal{L}_e$ is causal if

$$(Hu)_\tau = (Hu_\tau)_\tau$$

Def ~~#~~ 5.1 A mapping $H : \mathcal{L}_e \to \mathcal{L}_e$ is $\mathcal{L}$-stable if $\exists$ a class ~~of~~ $\mathcal{K}$ function α defined on $[0,\infty)$ and $\exists \beta > 0$ s.t.

$$\|(Hu)_\tau\|_{\mathcal{L}} \le \alpha(\|u_\tau\|_{\mathcal{L}}) + \beta$$

for all $u \in \mathcal{L}_e$, and $\forall \tau \in [0,\infty)$

It is finite gain stable if $\exists \gamma, \beta > 0$ s.t.

$$\|(Hu)_\tau\|_{\mathcal{L}} \le \gamma \|u_\tau\|_{\mathcal{L}} + \beta$$

$\forall u \in \mathcal{L}_e$ and $\forall \tau \in [0,\infty)$.

The smallest possible γ such that this inequality is satisfied is called the gain of the system.

Example: For the linear system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\gamma = \|Ce^{At}B\|_{\mathcal{L}_1} + \|D\| \quad , \beta = \|Ce^{At}x_0\|_{\mathcal{L}_\infty}$$

We can also define a local version of this result:

Def: ~~?~~ 5.2 H is small-signal $\mathcal{L}$ stable if

$$\|(Hu)_\tau\|_{\mathcal{L}} \leq \gamma \|u_\tau\|_{\mathcal{L}} + \beta$$

$$\forall u \text{ s.t. } \sup_{0 \leq t \leq \tau} \|u(t)\| \leq r < \infty .$$

~~6.1~~ 5.2 $\mathcal{L}$ stability of State models

$$\dot{x} = f(t, x, u)$$
$$y = h(t, x, u)$$

<u>Corollary ~~Thm~~ 5.1</u>

If 1) $x = 0$ is (globally) exponentially stable

2) f, h satisfy

$$\left.\begin{aligned} \|f(t,x,u) - f(t,x,0)\| &\le L\|u\| \\ \|h(t,x,u)\| &\le \eta_1(\|x\|) + \eta_2\|u\| \end{aligned}\right\}$$

true if Jacobians are locally bounded in t around origin

Then the system is small-signal, (globally) finite-gain $\mathcal{L}_p$ stable

<u>Example 6.5</u>

$$\dot{x} = -x - x^3 + u \qquad x(0) = x_0$$
$$y = \tanh(x) + u$$

Since $\dot{x} = -x - x^3$ is globally exponentially stable

and $\|f(t,x,u) - f(t,x,0)\| = \|u\|$

and $\|\tanh(x) + u\| \le \|x\| + \|u\|$

then the system is $\mathcal{L}_p$ stable for any $\mathcal{L}_p$.

Thm 5.3

If 1) $\dot{x} = f(t,x,u)$ is ISS

2) $\|h(t,x,u)\| \le \alpha_1(\|x\|) + \alpha_2(\|u\|) + \eta_3$

where $\alpha_1, \alpha_2 \in \mathcal{K}$, $\eta_3 > 0$

Then the system is $\mathcal{L}_\infty$ stable.

proof:

ISS $\Rightarrow$ $\|x(t)\| \le \beta(\|x(t_0)\|, t-t_0) + \gamma\left(\sup_{t_0 \le \tau \le t} \|u(\tau)\|\right)$

$\therefore$ $\|y(t)\| = \|h(t,x,u)\| \le \alpha_1(\|x(t)\|) + \alpha_2(\|u(t)\|) + \eta_3$

$\le \alpha_1\left(\beta(\|x(t_0)\|, t-t_0) + \gamma\left(\sup \|u(\tau)\|\right)\right) + \alpha_2(\|u(t)\|) + \eta_3$

Now note that if $\alpha \in \mathcal{K}$ and $a, b > 0$

$\alpha(a+b) \le \alpha(2\max(a,b))$

$\le \alpha(2\max(a,b)) + \alpha(2\min(a,b))$

$= \alpha(2a) + \alpha(2b)$

$\therefore$ $\|y(t)\| \le \alpha_1(2\beta(\|x(t_0)\|, t-t_0)) + \alpha_1\left(2\gamma\left(\sup \|u(\tau)\|\right)\right) + \alpha_2(\|u(t)\|) + \eta_3$

$\le \gamma_0(\|u_\tau\|_{\mathcal{L}_\infty}) + \beta_0$

when $\gamma_0 = \alpha_1 \circ 2\gamma + \alpha_2$

$\beta_0 = \alpha_1(2\beta(\|x(t_0)\|, 0)) + \eta_3$

Example 5.6

$$\dot{x} = -x + 2x^3 + (1+x^2)u^2$$
$$y = x^2 + u$$

We already showed that $\dot{x} = -x + 2x^3 + (1+x^2)u^2$ is ISS.

Also $\|h(t,x,u)\| \le \alpha_1(\|x\|) + \alpha_2(\|u\|)$

where $\alpha_1(r) = r^2$ and $\alpha_2(r) = r$

$\therefore$ The system is $\mathcal{L}_\infty$ stable.

Example 5.7

$$\dot{x}_1 = -x_1^3 + g(t)\, x_2$$
$$\dot{x}_2 = -g(t)\, x_1 - x_2^3 + u$$
$$y = x_1 + x_2$$

where $g(t)$ is continuous and bounded for all $t \ge 0$

Try $V = x_1^2 + x_2^2$, then

$$\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$
$$= 2x_1(-x_1^3 + g\, x_2) + 2x_2(-g\, x_1 - x_2^3 + u)$$
$$= -2x_1^4 + 2g\, x_1 x_2 - 2g\, x_1 x_2 - 2x_2^4 + 2x_2 u$$
$$= -2x_1^4 - 2x_2^4 + 2x_2 u$$

Since $\frac{1}{2}\|x\|_2^4 = \frac{1}{2}(x_1^2 + x_2^2)^2 = \frac{1}{2}x_1^4 + \underbrace{x_1^2 x_2^2 + \frac{1}{2}x_2^4}\; \le x_1^4 + x_2^4$

$$\Leftrightarrow \quad 0 \le \frac{1}{2}x_1^4 - x_1^2 x_2^2 + \frac{1}{2}x_2^4$$
$$\Leftrightarrow \quad 0 \le \frac{1}{2}(x_1^2 - x_2^2)^2$$

$\therefore \quad \dot{V} \le -\|x\|^4 + 2\|x\|\,|u|$

$= -(1-\theta)\|x\|^4 - \theta\|x\|^4 + 2\|x\|\,|u| \qquad 0<\theta<1$

$\le -(1-\theta)\|x\|^4 \quad \forall$

if $\quad -\theta\|x\|^4 + 2\|x\|\,|u| < 0$

$\Leftrightarrow \quad 2|u| < \theta\|x\|^3$

$\Leftrightarrow \quad \|x\| > \left(\dfrac{2|u|}{\theta}\right)^{1/3}$

$\therefore$ the state equation is ISS

Since $\quad y = x_1 + x_2 \le \sqrt{2}\,\|x\|$

$\therefore$ the system is $\mathcal{L}_\infty$ stable

(ie BIBO stable)

$\mathcal{L}_2$ Gain

- The $\mathcal{L}_2$ gain of ~~the~~ a system is the smallest γ s.t. $\exists \beta$ s.t.

$$\| y_\tau \|_{\mathcal{L}_2} \le \gamma \| u_\tau \|_{\mathcal{L}_2} + \beta$$

Thm 6.4

Given

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where A - Hurwitz. Let $G(s) = C(sI-A)^{-1}B + D$.

Then the $\mathcal{L}_2$ - gain is $\sup_{\omega \in \mathbb{R}} \| G(j\omega) \|_2$.

Proof:

Let $x(0) = 0$. We know that $Y(s) = G(s)U(s)$,

$$\Rightarrow \quad Y(j\omega) = G(j\omega)\, U(j\omega)$$

Parseval's theorem states that if $y \in \mathcal{L}_2$ $\int_0^\infty y^T(t) y(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y^*(j\omega) Y(j\omega)\, d\omega$.

$\therefore$

$$\| y \|_{\mathcal{L}_2}^2 = \int_0^\infty y^T(t)\, y(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y^*(j\omega)\, Y(j\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} U^*(j\omega)\, G^*(j\omega)\, G(j\omega)\, U(j\omega)\, d\omega$$

$$\le \left(\sup_{\omega \in \mathbb{R}} \| G(j\omega) \|_2 \right)^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} U^*(j\omega)\, U(j\omega)\, d\omega$$

$$= \left(\sup_{\omega \in \mathbb{R}} \| G(j\omega) \|_2 \right)^2 \| u \|_{\mathcal{L}_2}^2$$

Therefore $\gamma = \Big(\underbrace{\sup_{\omega \in \mathbb{R}} \| G(j\omega) \|_2}_{} \Big)$

if A-Hurwitz then this is called the $\mathcal{H}_\infty$-norm of the system.

__Thm ~~II~~ 5.5__

Consider

$$\dot{x} = f(x) + G(x)u \, , \qquad x(0) = x_0$$
$$y = h(x)$$

f - locally Lipschitz

G, h - continuous (why not locally Lipschitz ?)

$f(0) = 0$, $h(0) = 0$

If $\gamma > 0$ and $\exists$ cont. diff, p.d. V s.t.

$$\mathcal{H}(V, f, G, h, \gamma) \triangleq \frac{\partial V}{\partial x} f + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G G^T \frac{\partial V}{\partial x} + \frac{1}{2} h^T h \leq 0 \qquad \forall x \in \mathbb{R}^n$$

Then for each $x_0 \in \mathbb{R}^n$, the system is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain $= \gamma$.

~~proof~~ :

$$\dot{V} = \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial x} G u \leq \frac{\partial V}{\partial x} G u - \frac{1}{2} h^T h - \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G G^T \frac{\partial V}{\partial x}$$

$$\cancel{= -\frac{1}{2} \| y(t) \|_2^2 + \frac{\partial V}{\partial x} G \Big(u - \frac{1}{\gamma^2} G^T \frac{\partial V}{\partial x} \Big) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G G^T \frac{\partial V}{\partial x}}$$

$$\cancel{= -\frac{1}{2} \| y(t) \|_2^2 +}$$

$$= -\frac{1}{2} \| y \|_2^2 + \frac{\partial V}{\partial x} G u - \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G G^T \frac{\partial V}{\partial x}$$

Now notice that

$$-\frac{\gamma^2}{2}\left(u-\frac{1}{\gamma^2}G^T\frac{\partial V}{\partial x}^T\right)^T\left(u-\frac{1}{\gamma^2}G^T\frac{\partial V}{\partial x}^T\right)$$

$$= -\frac{\gamma^2}{2}u^Tu + \frac{\partial V}{\partial x}Gu - \frac{1}{2\gamma^2}\frac{\partial V}{\partial x}GG^T\frac{\partial V}{\partial x}^T$$

$\therefore$

$$\dot V = -\frac{1}{2}\|y\|_2^2 + \frac{\gamma^2}{2}u^Tu - \frac{\gamma^2}{2}\left\|u-\frac{1}{\gamma^2}G^T\frac{\partial V}{\partial x}^T\right\|^2$$

$$\leq \frac{\gamma^2}{2}\|u\|_2^2 - \frac{1}{2}\|y\|_2^2$$

$\therefore$ integrating along trajectories of the system gives

$$\int_0^\tau \dot V(t)\,dt = V(x(\tau)) - V(x_0) \leq \frac{\gamma^2}{2}\int_0^\tau \|u(t)\|_2^2\,dt - \frac{1}{2}\int_0^\tau \|y(t)\|_2^2\,dt$$

or $$0 \leq V(x(\tau)) \leq V(x_0) + \frac{\gamma^2}{2}\|u_\tau\|_{\mathcal{L}_2}^2 - \frac{1}{2}\|y_\tau\|_{\mathcal{L}_2}^2$$

$$\Rightarrow \quad \|y_\tau\|_{\mathcal{L}_2}^2 \leq \gamma^2\|u_\tau\|_{\mathcal{L}_2}^2 + 2V(x_0)$$

~~thy~~ since $\sqrt{a^2+b^2} \leq a+b$

$$\|y_\tau\|_{\mathcal{L}_2} \leq \gamma\|u_\tau\|_{\mathcal{L}_2} + \underbrace{\sqrt{2V(x_0)}}_{\beta}$$

example ~~Ex~~ 5.8

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a x_1^3 - k x_2 + u \qquad a, k > 0$$
$$y = x_2$$

Let $V = \alpha \left(\frac{1}{4} a x_1^4 + \frac{1}{2} x_2^2 \right) \qquad \alpha > 0$

$$\frac{\partial V}{\partial x}^T = \begin{pmatrix} \alpha a x_1^3 \\ \alpha x_2 \end{pmatrix}, \quad f = \begin{pmatrix} x_2 \\ -a x_1^3 - k x_2 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad h = x_2$$

$\therefore$
$$\frac{\partial V}{\partial x} f + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} g g^T \frac{\partial V}{\partial x}^T + \frac{1}{2} h^T h$$

$$= \alpha a x_1^3 x_2 - \alpha a x_1^3 x_2 - \alpha k x_2^2 + \frac{1}{2\gamma^2} \alpha^2 x_2^2 + \frac{1}{2} x_2^2$$

$$= -\alpha k x_2^2 + \frac{\alpha^2}{2\gamma^2} x_2^2 + \frac{1}{2} x_2^2$$

$\therefore$ need to find $\alpha > 0$ and $\gamma > 0$ s.t.

$$-\alpha k + \frac{\alpha^2}{2\gamma^2} + \frac{1}{2} \leq 0$$

$$\Leftrightarrow \quad \frac{\alpha^2}{\gamma^2} \leq 2\alpha k - 1$$

$$\Leftrightarrow \quad \frac{\gamma^2}{\alpha^2} \geq \frac{1}{2\alpha k - 1}$$

$$\Leftrightarrow \quad \gamma^2 \geq \frac{\alpha^2}{2\alpha k - 1}$$

Find α to minimize RHS $\qquad \frac{\partial}{\partial \alpha} \left(\frac{\alpha^2}{2\alpha k - 1} \right) = \frac{(2\alpha k - 1) 2\alpha - \alpha^2 2k}{(2\alpha k - 1)^2} = 0$

$$\Leftrightarrow \quad 4\alpha^2 k - 2\alpha - 2\alpha^2 k = 0 \quad \Leftrightarrow$$
$$\Leftrightarrow \quad 2\alpha k = 2$$
$$\Leftrightarrow \quad \boxed{\alpha = \frac{1}{k}} \quad \Leftrightarrow \quad \gamma^2 \geq \frac{\frac{1}{k^2}}{2-1} = \frac{1}{k^2} \quad \Leftrightarrow \quad \boxed{\gamma \geq \frac{1}{k}}$$

Example 6.11

Consider $\dot{x} = f(x) + G(x)u$

$y = h(x)$

Suppose there exists a cont. diff, pd function W s.t.

$$\frac{\partial W}{\partial x} f \le 0$$

$$\frac{\partial W}{\partial x} G = h^T$$

Then let $u = -ky + v \quad k > 0$

ie $\dot{x} = \left[f(x) - k G G^T \frac{\partial W}{\partial x}^T\right] + Gv \triangleq f_c + Gv$

$y = h = G^T \frac{\partial W}{\partial x}^T$

Then let $V = \alpha W$

$$\frac{\partial V}{\partial x} f_c + \frac{1}{2} h^T h + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G G^T \frac{\partial V}{\partial x}$$

$$= \alpha \frac{\partial W}{\partial x} f - \alpha k \frac{\partial W}{\partial x} G G^T \frac{\partial W}{\partial x}^T + \frac{1}{2} \frac{\partial W}{\partial x} G G^T \frac{\partial W}{\partial x}^T + \frac{\alpha^2}{2\gamma^2} \frac{\partial W}{\partial x} G G^T \frac{\partial W}{\partial x}$$

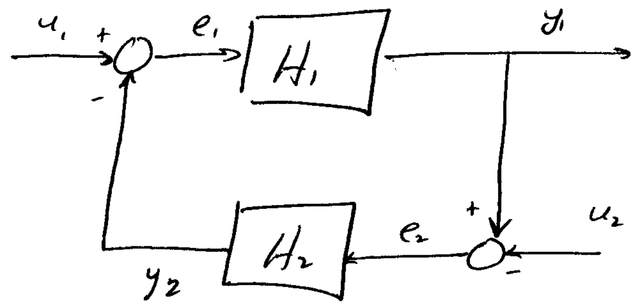
$$\le \left(-\alpha k + \frac{1}{2} + \frac{\alpha^2}{2\gamma^2}\right) \frac{\partial W}{\partial x} G G^T \frac{\partial W}{\partial x}$$

$\therefore$ $\mathcal{L}_2$ gain is $\gamma = \frac{1}{k}$ and we can make γ as small as we would like via high gain ~~contr~~ output feedback !!

10.2 Small-Gain Theorem

~~On the first exam we saw~~

Consider the feedback system



On the first exam we saw that if $\gamma_1 \gamma_2 < 1$ then if $u_1, u_2 \in \mathcal{L}$ for some Banach space $\mathcal{L}$, then, e_1, e_2 (and hence y_1, y_2) had unique solutions.

We now turn to the question of stability

~~Thm 10.5 (Small-Gain theorem)~~
~~If $\gamma_1 \gamma_2 < 1$, then for all u_1~~

Suppose $H_1 : \mathcal{L}_e^m \to \mathcal{L}_e^q$ and
$H_2 : \mathcal{L}_e^q \to \mathcal{L}_e^m$

and suppose both H_1, H_2 are finite-gain $\mathcal{L}$ stable, ie

$$\| y_{1\tau} \|_{\mathcal{L}} \leq \gamma_1 \| e_{1\tau} \|_{\mathcal{L}} + \beta_1, \quad \forall e_1 \in \mathcal{L}_e^m, \ \forall \tau \geq 0$$

$$\| y_{2\tau} \|_{\mathcal{L}} \leq \gamma_2 \| e_{2\tau} \|_{\mathcal{L}} + \beta, \quad \forall e_2 \in \mathcal{L}_e^q, \ \forall \tau \geq 0$$

Thm 10.5 Under the preceding assumptions, if $\gamma_1 \gamma_2 < 1$, then $\forall\, u_1, u_2 \in \mathcal{L}_e$

$$\| e_{1\tau} \|_{\mathcal{L}} \le \frac{1}{1-\gamma_1\gamma_2} \left(\|u_{1\tau}\|_{\mathcal{L}} + \gamma_2 \|u_{2\tau}\|_{\mathcal{L}} + \beta_2 + \gamma_2 \beta_1 \right)$$

$$\| e_{2\tau} \|_{\mathcal{L}} \le \frac{1}{1-\gamma_1\gamma_2} \left(\|u_{2\tau}\|_{\mathcal{L}} + \gamma_1 \|u_{1\tau}\|_{\mathcal{L}} + \beta_1 + \gamma_1 \beta_2 \right)$$

proof:

$$e_{1\tau} = u_{1\tau} - (H_2[e_2])_\tau = u_{1\tau} - y_{2\tau}$$

$$e_{2\tau} = u_{2\tau} + (H_1 e_1)_\tau = u_{2\tau} + y_{1\tau}$$

$\Rightarrow$

$$\|e_{1\tau}\|_{\mathcal{L}} \le \|u_{1\tau}\|_{\mathcal{L}} + \|y_{2\tau}\|$$

$$\le \|u_{1\tau}\|_{\mathcal{L}} + \gamma_2 \|e_{2\tau}\|_{\mathcal{L}} + \beta_2$$

Also

$$\|e_{2\tau}\|_{\mathcal{L}} \le \|u_{2\tau}\|_{\mathcal{L}} + \gamma_1 \|e_{1\tau}\|_{\mathcal{L}} + \beta_1$$

$\Rightarrow$

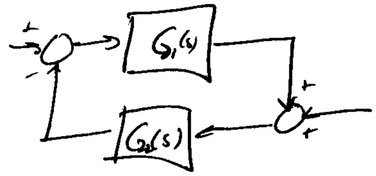
$$\|e_{1\tau}\|_{\mathcal{L}} \le \|u_{1\tau}\|_{\mathcal{L}} + \gamma_2 \|u_{2\tau}\|_{\mathcal{L}} + \gamma_1\gamma_2 \|e_{1\tau}\|_{\mathcal{L}} + \gamma_2\beta_1 + \beta_2$$

$\Rightarrow$

$$\|e_{1\tau}\|_{\mathcal{L}} \le \frac{1}{1-\gamma_1\gamma_2} \left[\|u_{1\tau}\|_{\mathcal{L}} + \gamma_2 \|u_{2\tau}\|_{\mathcal{L}} + \beta_2 + \gamma_2\beta_1 \right]$$

Similarly for $\|e_{2\tau}\|_{\mathcal{L}}$.

Example: Consider the feedback connection of two ~~linear~~ stable linear systems



Then the small gain theorem requires

$$(*) \qquad \left(\sup_\omega \|G_1(j\omega)\|\right)\left(\sup_\omega \|G_2(j\omega)\|\right) < 1$$

Noting that

$$\left(\sup_\omega \|G_1(j\omega)\|\right)\left(\sup_\omega \|G_2(j\omega)\|\right) = \sup_\omega \sup_\sigma \|G_1(j\omega)\|\|G_2(j\sigma)\|$$

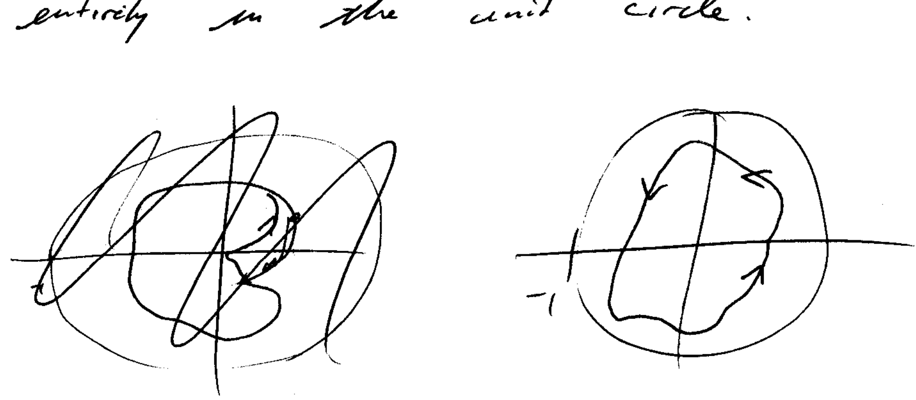
$$\geq \sup_\omega \|G_1(j\omega)\|\|G_2(j\omega)\|$$

$$\geq \sup_\omega \|G_1(j\omega)G_2(j\omega)\|$$

We see that if $(*)$ holds then of necessity

$$\sup_\omega \|G_1(j\omega)G_2(j\omega)\| < 1$$

Which is another way of saying that the Nyquist plot of $G_1(s)G_2(s)$ lies entirely in the unit circle.



Clearly the small gain theorem is extremely conservative!!