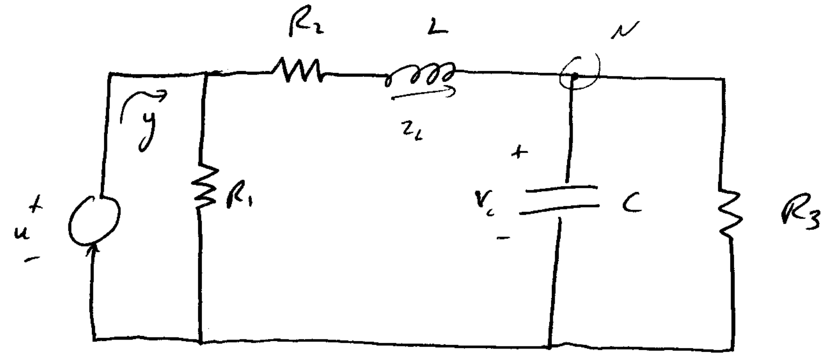
10.3 Passivity

Consider the LRC circuit



$$v_L = L\frac{di_L}{dt}$$
$$i_C = C\frac{dv_C}{dt}$$

Let $x_1 = i_L$

$x_2 = v_C$

~~Writing the~~ Summing the voltage around the middle loop gives

$$u = i_L R_2 + V_L + V_C = R_2 x_1 + L\frac{dx_1}{dt} + x_2$$

$$\Rightarrow \quad L\dot{x}_1 = u - R_2 x_1 - x_2$$

Summing the currents at node N gives

$$i_L = i_c + \frac{V_c}{R_3} \Rightarrow x_1 = C\dot{x}_2 + \frac{x_2}{R_3}$$

$$\Rightarrow \quad C\dot{x}_2 = x_1 - \frac{1}{R_3}x_2$$

The output y is given by

$$y = \frac{u}{R_1} + i_L = \frac{u}{R_1} + x_1$$

∴ the circuit model is

$$L\dot{x}_1 = u - R_2 x_1 - x_2$$
$$C\dot{x}_2 = x_1 - \frac{1}{R_3} x_2$$
$$y = x_1 + \frac{1}{R_1} u$$

The stored energy in the circuit at ~~and~~ any time t, is given by

$$V(x(t)) = \frac{1}{2} L x_1^2(t) + \frac{1}{2} C x_2^2(t)$$

Since the circuit is passive, the energy stored in the circuit must be less than or equal to the energy flowing into the circuit (the rest is dissipated in the resistors).

The instantaneous power flowing ~~out~~ into the network is $v(t) z(t) = u(t) y(t)$

therefore the energy ~~is~~ flowing ~~in~~ is

$$\int_0^t u(s) y(s)\, ds$$

The passivity of the circuit requires that

$$\int_0^t u(s) y(s)\, ds \geq V(x(t)) - V(x(0))$$

Alternatively we could have written

$$u(t) y(t) \geq \dot{V}(x(t))$$

ie the power flowing into the circuit is greater than the instantaneous change in stored energy.

For the circuit

$$\dot V = L x_1 \dot x_1 + C x_2 \dot x_2$$

$$= u x_1 - R_2 x_1^2 - \cancel{x_1 x_2} + \cancel{x_1 x_2} - \frac{1}{R_3} x_2^2 + uy - uy$$

$$= uy - R_2 x_1^2 - \frac{1}{R_1} x_2^2 + u x_1 - u x_1 - \frac{1}{R_1} u^2$$

$$= uy - R_2 x_1^2 - \frac{1}{R_1} x_2^2 - \frac{1}{R_1} u^2$$

$$\therefore \quad uy = \dot V + R_2 x_1^2 + \frac{1}{R_1} x_2^2 + \frac{1}{R_1} u^2$$

$$\Rightarrow \quad uy \geq \dot V$$

which verifies our intuition.

Note that the dissipation of energy will be equal to the difference between uy and $\dot V$

<u>Case I</u> $R_1 = R_3 = \infty$, $R_2 = 0$ (ie no current flows through the resistors, then

$$\dot V = uy$$

and there is no energy dissipation, ~~ie the system is "lossless"~~

we will call such systems "lossless"

Case 2 $\quad R_2 = 0 \quad R_3 = \infty$

$$uy = \dot{V} + \frac{1}{R_1} u^2$$

ie the dissipation is proportional to u^2.

no dissipation iff $u = 0$

We will call such systems

"input strictly passive"

Case 3 $\quad R_1 = R_3 = \infty$:

$$uy = \dot{V} + R_2 x_1^2 = \dot{V} + R_2 y^2 \quad \left(y = x_1 + \tfrac{1}{R_1} u = x_1 \right)$$

∴ dissipation proportional to output power : $0 \Leftrightarrow y = 0$

We will call such systems

"output strictly passive"

Case 4 $\quad R_1 = \infty$

$$uy = \dot{V} + R_2 x_1^2 + \frac{1}{R_3} x_2^2$$ ← pd function of x

∴ dissipation $= 0 \Leftrightarrow x = 0$

Such systems are called

"state strictly passive"

Case 5 $\quad R_1 = \infty, R_2 = 0$

$$uy = \dot{V} + \frac{1}{R_3} x_2^2$$

psd function of x

but $x_2 \equiv 0 \Rightarrow x_1 = 0$ from state equations

∴ again dissipation $= 0 \Leftrightarrow x = 0$.

Consider the system

$$\dot{x} = f(x,u)$$
$$y = h(x,u)$$

where $u \in \mathbb{R}^m$ <u>and</u> $y \in \mathbb{R}^m$

(ie same # of inputs as outputs)

f - locally Lipschitz

h - cont.

$f(0,0) = 0$, $h(0,0) = 0$

~~Def~~ Let $\psi(x)$ is a psd function such that

$$\psi(x(t)) \equiv 0 \Rightarrow x(t) \equiv 0 \quad \forall (x,y)$$

<u>Def 10.4</u> The system is said to be <u>passive</u> if $\exists$ cont. diff pd V (called the storage function) s.t.

$$u^T y \geq \frac{\partial V}{\partial x} f(x,u) + \varepsilon u^T u + \delta y^T y + \rho \psi(x)$$

where, $\varepsilon, \delta, \rho \geq 0$.

The system is called

- "<u>lossless</u>" if $\varepsilon = \delta = \rho = 0$ (ie $u^T y = \frac{\partial V}{\partial x} f$)
- "~~input strictly passive~~" if $\varepsilon > 0$
- "<u>output strictly passive</u>" if $\delta > 0$
- "<u>state strictly passive</u>" if $\rho > 0$

<u>Def 10.5</u> The system is <u>zero-state</u> observable if no solution of $\dot{x} = f(x, \underset{u=0}{0})$ can stay in $S = \{x \in \mathbb{R}^n \mid h(x,0) = 0\}$ other than the trivial solution.

Lemma 10.6

I. If $\left(\begin{array}{l}\dot{x}=f(x,u)\\ y=h(x,u)\end{array}\right)$ is passive with pd storage function $V(x)$, then the origin of $\dot{x}=f(x,0)$ is stable

proof: Since V - pd, it is a Lyapunov function candidate. Passivity gives $\dot{V} \le u^T y$. If $u=0$, then $\dot{V}\le 0$, therefore $\dot{x}=f(x,0)$ is stable.

II. If the system is output-strictly passive, then it is finite-gain $\mathcal{L}_2$ stable.

proof: Output-strictly passive implies that

$$\dot{V} \le u^T y - \delta y^T y$$

Completing the square, we get

$$\dot{V} \le u^T y - \delta y^T y = -\frac{1}{2\delta}(u-\delta y)^T(u-\delta y) + \frac{1}{2\delta}u^T u - \frac{\delta}{2}y^T y$$

$$\le \frac{1}{2\delta}u^T u - \frac{\delta}{2}y^T y$$

Integrating both sides gives

$$0 \le V(x(t)) \le V(x(0)) + \frac{1}{2\delta}\int_0^t u^T(\tau)u(\tau)\,d\tau - \frac{\delta}{2}\int_0^t y^T(\tau)y(\tau)\,d\tau$$

$$\therefore \quad \frac{\delta}{2}\|y_\tau\|^2_{\mathcal{L}_2} \le \frac{1}{2\delta}\|u_\tau\|^2_{\mathcal{L}_2} + V(x(0))$$

$$\Rightarrow \quad \|y_\tau\|_{\mathcal{L}_2} \le \frac{1}{\delta}\|u_\tau\|_{\mathcal{L}_2} + \sqrt{\frac{2}{\delta}V(x(0))}$$

$$\therefore \quad \mathcal{L}_2\text{-gain} \le \frac{1}{\delta}$$

III If the system is output-strictly passive, with pd storage function, and zero-state observable, then the origin of $\dot{x} = f(x,0)$ is asymptotically stable

<u>proof</u>: Output-strictly passive $\Rightarrow$

$$\dot{V} = \frac{\partial V}{\partial x} f(x,u) \le u^T y - \delta h^T(x,u)\, h(x,u)$$

$$\Rightarrow \dot{V}\Big|_{u=0} = \frac{\partial V}{\partial x} f(x,0) \le -\delta h^T(x,0)\, h(x,0) \le 0$$

therefore $\dot{V}$ is nsd.

Let $M = \{x : h(x,0) = 0\}$,

Zero-state observability implies that the only invariant set in M is $\{x = 0\}$. $\therefore$ By LaSalle's theorem, the origin is asymptotically stable.

IV. If the system is state-strictly passive with pd storage function V, then the origin of $\dot{x} = f(x,u)$ is asymptotically stable.

proof: same as above.

Example 10.12 Consider the system

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

Suppose there exits a ~~pd~~ psd $V(x)$ s.t.

$$\frac{\partial V}{\partial x} f \le 0$$
$$\frac{\partial V}{\partial x} g = h^T$$

Then the system is passive since

$$u^T y - \frac{\partial V}{\partial x}(f+gu) = u^T h - \frac{\partial V}{\partial x} f - h^T u = -\frac{\partial V}{\partial x} f \ge 0$$

$$\Rightarrow \quad u^T y \ge \frac{\partial V}{\partial x}(f+gu).$$

Suppose $\exists$ a psd V s.t.

$$\frac{\partial V}{\partial x} f \le -k h^T h, \quad k>0$$
$$\frac{\partial V}{\partial x} g = h^T$$

Then the system is output-strictly passive since.

$$u^T y - \frac{\partial V}{\partial x}(f+gu) = u^T h - \frac{\partial V}{\partial x} f - h^T u \ge +k h^T h$$
$$= +k y^T y$$

$$\Rightarrow \quad \dot{V} \le u^T y - k y^T y$$

Memoryless (ie no states) system

Consider

$$y = h(t,u) \quad \text{where}$$

h - piecewise cont. in t, locally Lipschitz in u.

Def 10.6 The system $y = h(t,x)$ is passive if

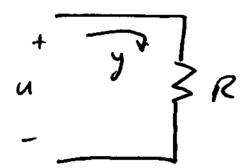
$$u^T y \geq \varepsilon u^T u + \delta y^T y$$

where $\varepsilon, \delta > 0$.

If $\delta > 0$ the system is output-strictly passive

If $\varepsilon > 0$ the system is input-strictly passive.

Example 10.13



Since $u = yR$ we have

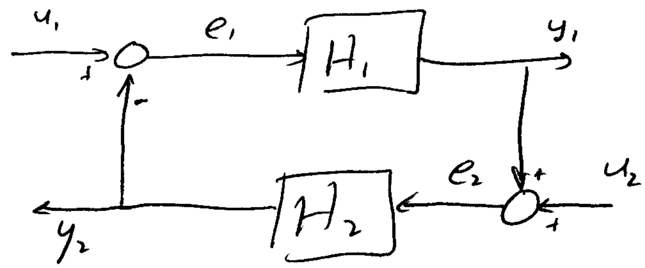
$$uy = Ry^2 = \tfrac{1}{R}u^2$$

$$= \tfrac{1}{2}uy + \tfrac{1}{2}uy = \tfrac{R}{2}y^2 + \tfrac{1}{2R}u^2$$

$\therefore$ a resistor is both output-strictly passive and input-strictly passive.

Note that Def 10.6 is the same as Def 10.4 if we let $V = 0$.

Consider the feedback system



where we will assume that unique solutions exist,
ie the feedback loop is well-posed.

Also assume that H_1 and H_2 are passive
ie

$$e_i^T y_i \geq \frac{\partial V_i}{\partial x_i} f_i(x_i, e_i) + \varepsilon_i e_i^T e_i + \delta_i y_i^T y_i \qquad i=1,2$$

<u>Thm 10.6</u> The map from $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \longmapsto \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ is finite-gain $\mathcal{L}_2$-stable if

$$\varepsilon_1 + \delta_2 > 0 \quad \text{and} \quad \varepsilon_2 + \delta_1 > 0$$

<u>proof:</u> Let $V = V_1 + V_2$, $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

then
$$\frac{\partial V}{\partial x} f = \frac{\partial V_1}{\partial x} f_1 + \frac{\partial V_2}{\partial x} f_2$$
$$\leq e_1^T y_1 - \varepsilon_1 e_1^T e_1 - \delta_1 y_1^T y_1 + e_2^T y_2 - \varepsilon_2 e_2^T e_2 - \delta_2 y_2^T y_2$$

Lets get all e_i's in terms of u_i's + y_i's:

$$e_1 = u_1 - y_2, \qquad e_2 = y_1 + u_2$$

$$\Rightarrow \frac{\partial V}{\partial x} f \leq u_1^T y_1 - y_2^T y_1 - \varepsilon_1 u_1^T u_1 + 2\varepsilon_1 u_1^T y_2 - \varepsilon_1 y_2^T y_2 - \delta_1 y_1^T y_1$$
$$+ y_1^T y_2 + u_2^T y_2 - \varepsilon_2 y_1^T y_1 - 2\varepsilon_2 y_1^T u_2 - \varepsilon_2 u_2^T u_2 - \delta_2 y_2^T y_2$$

$$\therefore \frac{\partial V}{\partial x} f \leq -\begin{pmatrix} y_1^T & y_2^T \end{pmatrix} \underbrace{\begin{pmatrix} (\delta_1+\varepsilon_2)I & 0 \\ 0 & I(\delta_2+\varepsilon_1) \end{pmatrix}}_{\triangleq L} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$- \begin{pmatrix} u_1^T & u_2^T \end{pmatrix} \underbrace{\begin{pmatrix} \varepsilon_1 I & 0 \\ 0 & \varepsilon_2 I \end{pmatrix}}_{\triangleq M} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$+ \begin{pmatrix} u_1^T & u_2^T \end{pmatrix} \underbrace{\begin{pmatrix} I & 2\varepsilon_1 I \\ -2\varepsilon_2 I & I \end{pmatrix}}_{\triangleq N} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\therefore \frac{\partial V}{\partial x} f \leq -y^T L y - u^T M u + u^T N y$$

By ~~assumption~~ hypothesis $L > 0$ and $M \geq 0$

$$\therefore \dot{V} \leq -\underline{\sigma}(L) \|y\|_2^2 - \underline{\sigma}(M) \|u\|_2^2 + \|u\|_2 \|y\|_2 \overset{\bar{\sigma}(N)}{\overset{\shortparallel}{\|N\|_2}}$$

$$\leq -\underline{\sigma}(L) \|y\|_2^2 + \|u\|_2 \|y\|_2 \|N\|_2$$

$$= -\frac{1}{2\underline{\sigma}(L)} \left(\|N\|_2 \|u\|_2 - \underline{\sigma}(L) \|y\|_2 \right)^2 + \frac{\|N\|_2^2}{2\underline{\sigma}(L)} \|u\|_2^2 - \frac{\underline{\sigma}(L)}{2} \|y\|_2^2$$

$$\leq \frac{\|N\|_2^2}{2\underline{\sigma}(L)} \|u\|_2^2 - \frac{\underline{\sigma}(L)}{2} \|y\|_2^2$$

$$\Rightarrow \|y_\tau\|_{\mathcal{L}_2} \leq \frac{\|N\|_2}{\underline{\sigma}(L)} \|u_\tau\|_{\mathcal{L}_2} + \sqrt{\frac{2}{\underline{\sigma}(L)} V(x(0))}$$

$$\therefore \text{the } \mathcal{L}_2 \text{ gain} \leq \frac{\bar{\sigma}(N)}{\underline{\sigma}(L)}$$

Thm 10.7 Suppose that H_1 + H_2 are passive,

ie

$$e_i^T y_i \geq \frac{\partial V_i}{\partial x_i} f_i(x_i, e_i) + \varepsilon e_i^T e_i + \delta y_i^T y_i + \rho \psi_i(x_i)$$

+ suppose ~~there~~ is a well defined state space model

$$\dot{x} = f(x, u)$$

~~where~~ $y = h(x, u)$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Then $\dot{x} = f(x, 0)$ is asymptotically stable in any of the following cases

I. $\rho_1 > 0$ and $\rho_2 > 0$

II. $\rho_1 > 0$, $\varepsilon_1 + \delta_2 > 0$, H_2 - zero-state observable

III. $\rho_2 > 0$, $\varepsilon_2 + \delta_1 > 0$, H_1 - zero state obse

IV $\varepsilon_1 + \delta_2 > 0$, $\varepsilon_2 + \delta_1 > 0$, both H_1, H_2 zero-state os.

proof: let $V = V_1 + V_2$, then

$$\dot{V} = \frac{\partial V_1}{\partial x_1} f_1(x_1, e_1) + \frac{\partial V_2}{\partial x_2} f_2(x_2, e_2)$$

$$\leq \sum_{i=1}^{2} \left[e_i^T y_i - \varepsilon_i e_i^T e_i - \delta_i y_i^T y_i - \rho_i \psi_i(x_i) \right]$$

$$= (u_1 - y_2)^T y_1 - \varepsilon_1 (u_1 - y_2)^T (u_1 - y_2) - \delta_1 y_1^T y_1 - \rho_1 \psi_1(x_1)$$
$$+ (u_2 + y_1)^T y_2 - \varepsilon_2 (u_2 + y_1)^T (u_2 + y_1) - \delta_2 y_2^T y_2 - \rho_2 \psi_2(x_2)$$

$$= u_1^T y_1 - y_2^T y_1 - \varepsilon_1 u_1^T u_1 + 2\varepsilon_1 u_1^T y_2 - \varepsilon_1 y_2^T y_2 - \delta_1 y_1^T y_1$$
$$- \rho_1 \psi_1(x_1) + u_2^T y_2 + y_1^T y_2 - \varepsilon_2 u_2^T u_2 - 2\varepsilon_2 u_2^T y_1 - \varepsilon_2 y_1^T y_1$$
$$- \delta_2 y_2^T y_2 - \rho_2 \psi_2(x_2)$$

Letting ~~u_1,u_2~~ $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$ gives

$$\dot{V} \leq -(\varepsilon_1+\delta_2)\, y_2^T y_2 - (\delta_1+\varepsilon_2)\, y_1^T y_1 - \rho_1 \psi_1(x_1) - \rho_2 \psi_2(x_2)$$

Case 1 : $\rho_1 > 0$, $\rho_2 > 0$ and the fact that

$\left.\begin{array}{l} \psi_1(x_1(t)) \equiv 0 \iff x_1(t) \equiv 0 \\ \psi_2(x_2(t)) \equiv 0 \iff x_2(t) \equiv 0 \end{array}\right\}$ implies asymptotic stability by

LaSalle's invariance principle

Case 2 : $\rho_1 > 0$, $\varepsilon_1 + \delta_2 > 0$, H_2 - zero - state observable

$\psi_1(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0$

$y_2^T y_2 \equiv 0$ + H_2-observable $\Rightarrow$ $x_2(t) \equiv 0$

etc.

Example : Mechanical Systems

Let the kinetic energy of a mechanical system be given by K and the potential energy given by P. Let $q = (q_1, \dots, q_n)^T$ be generalized configuration coordinates, and let

$$L(q, \dot{q}) = K(q, \dot{q}) - P(q, \dot{q})$$

and let $\tau = (\tau_1, \dots, \tau_n)^T$ be generalized forces acting on the system.

Then the equations of motion are given by the Euler - Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau$$

Alternatively, let $H = K + P$ be the total stored energy, then the Euler-Lagrange equations can be written as the Hamiltonian system

$$\dot{q} = \frac{\partial H}{\partial p}(q, p)$$

$$\dot{p} = \frac{\partial H}{\partial q}(q, p) + \tau \qquad (p = \text{generalized momentum})$$

Note that

$$\begin{aligned} \dot{H} &= \frac{\partial H}{\partial q}^T \dot{q} + \frac{\partial H}{\partial p}^T \dot{p} \\ &= \frac{\partial H}{\partial q}^T \frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}^T \frac{\partial H}{\partial q} + \frac{\partial H}{\partial p}^T \tau \\ &= \dot{q}\,\tau \end{aligned}$$

Therefore, the output $y = \dot{q}$ is a passive output.

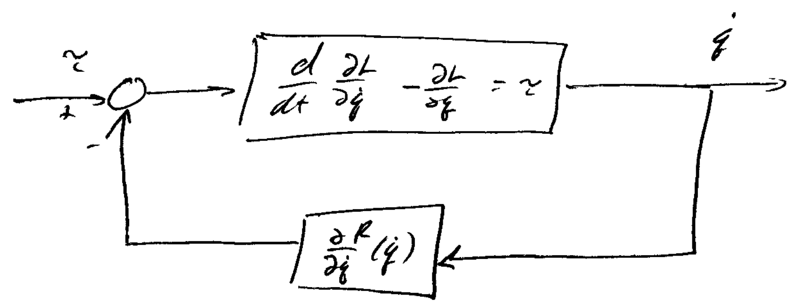
~~Th~~ If the system has friction, or any kind of internal dissipation, it can usually be modeled as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} + \frac{\partial R}{\partial \dot q}(\dot q) = \tau$$

where $R(\dot q)$ is the Rayleigh dissipation function satisfying

$$\dot q^T \frac{\partial R}{\partial \dot q} \geq 0 .$$

We can ~~write~~ interpret this function ~~as feedb~~ as a feedback interconnection:



Therefore the closed-loop system is passive.

If $\quad \dot q^T \frac{\partial R}{\partial \dot q}(\dot q) \geq \delta \|\dot q\|^2$

then the closed-loop ~~ct~~ system is ~~fi~~ finite-gain $\mathcal{L}_2$-stable

and $\tau = 0 \Rightarrow (q, \dot q) \to 0$