Midterm Exam
ECEn 774 Nonlinear System Theory, Winter 2010

Name: ______________________________

Open book, open notes, open computer, open time, open brain.
Closed neighbor, closed friend, closed library, closed other textbooks, closed internet search.

Exam is due at the beginning of class on March 1. Late Exams will not be accepted.

All questions concerning the exam must be sent to me via email (beard@byu.edu). I will send the question and my response to the class email list.

Problem 1	________	/ 25
Problem 2	________	/ 25
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Total	________	/ 100

1. (25 pts) Consider the following nonlinear system which defines a trajectory tracker for a UAV:

$$\begin{aligned} \dot{y} &= \sin(\psi) \\ \dot{\psi} &= \text{ sat } \left[-\tan^{-1}(k_1 y) - k_2 \sin(\psi)\right], \end{aligned}$$

where k_1 and k_2 are positive constants and where sat is the saturation function defined as

$$\text{sat}[\xi] = \begin{cases} 1, & \xi \geq 1 \\ -1, & \xi \leq -1 \\ \xi, & \text{otherwise} \end{cases} .$$

(a) Find the equilibria of the system.

(b) Using linearization, characterize each equilibria as one of the following: stable, asymptotically stable, unstable, cannot be determined from linearization.

(c) Using Matlab, plot the phase portrait for the system when $k_1 = k_2 = 1$. Explain the behavior of the system from the phase portrait.

2. (25 pts) Consider the full 6-DOF equations of motion for a quadrotor which are given by

$$\begin{pmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\phi\sin\theta & \cos\phi\sin\theta \\ 0 & \cos\phi & -\sin\phi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} rv - qw \\ pw - ru \\ qu - pv \end{pmatrix} + \begin{pmatrix} -g\sin\theta \\ g\cos\theta\sin\phi \\ g\cos\theta\cos\phi \end{pmatrix} + \frac{1}{m}\begin{pmatrix} 0 \\ 0 \\ -F \end{pmatrix},$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\cos\theta} \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix},$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{J_y - J_z}{J_x} qr \\ \frac{J_z - J_x}{J_y} pr \\ \frac{J_x - J_y}{J_z} pq \end{pmatrix} + \begin{pmatrix} \frac{1}{J_x}\tau_\phi \\ \frac{1}{J_y}\tau_\theta \\ \frac{1}{J_z}\tau_\psi \end{pmatrix},$$

where (p_x, p_y, p_z) are relative positions in the yawed vehicle frame, (u, v, w) are body frame velocities, (ϕ, θ, ψ) are the roll, pitch, and yaw angles respectively, (p, q, r) are the angular rates in the body frame, m is the mass, g is the gravitational constant, (J_x, J_y, J_z) are the moments of inertia about each axis, F is the commanded thrust, and $(\tau_\phi, \tau_\theta, \tau_\psi)$ are commanded torques.

If $F = \tau_\phi = \tau_\theta = \tau_\psi = 0$, are the resulting equations (1) continuous?, (2) continuously differentiable? (3) locally Lipschitz? (4) globally Lipshitz? Rigorously justify your answers. If the answer is yes over a certain domain D, precisely specify D.

3. (25 pts) The development in the book is limited to continuous-time systems. In this problem we will explore some extensions to nonlinear discrete-time systems given by

$$x[k+1] = f(x[k]), \quad x[0] = x_0, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$.

(a) Existence and Uniqueness. Derive sufficient conditions to guarantee that there exists a unique solution to Eq. (1).

(b) Stability. Provide suitable definitions of stability and asymptotic stability for discrete-time systems.

(c) Lyapunov Theory. Lyapunov theory for discrete-time systems is based on the rate of decrease of V at each time step, i.e., we are concerned with $\Delta V[k] = V(x[k+1]) - V(x[k])$ instead of $\dot{V}$. State and prove a Lyapunov-like result for discrete-time systems.

(d) Demonstrate the application of your result for the simple system $x[k+1] = ax[k]$ where $-1 < a < 1$, which is well-known to converge to zero.

4. (25 pts) **(a)** Work Exercise 1.4 in the book.

Many holonomic mechanical systems can be written in the form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u. \tag{2}$$

(b) Letting $q = \theta$ show that the inverted pendulum and the mass-spring-damper system can both be put into this form.

The total energy for system Eq (2) can be expressed as

$$E(q,\dot{q}) = P(q) + \frac{1}{2}\dot{q}^T M(q)\dot{q}. \tag{3}$$

(c) If $u = 0$, show that the total energy of the system is non-increasing.

A PD control law for system (2) can be expressed as

$$u_{PD} = -K_p q - K_d \dot{q}.$$

(d) If $u = u_{PD}$ and $P(q) = \frac{1}{2}q^T S q$ where $S = S^T > 0$, use the Lasalle's invariance principle to argue that if $K_p = K_p^T > 0$ and $K_d = K_d^T > 0$, then $(q,\dot{q}) \to 0$. Hint: Use the Lyapunov function $V = E + \frac{1}{2}q^T K_p q$