Kinematics for a Three Wheeled Mobile Robot

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1 Reference Frames and 2D Rotations

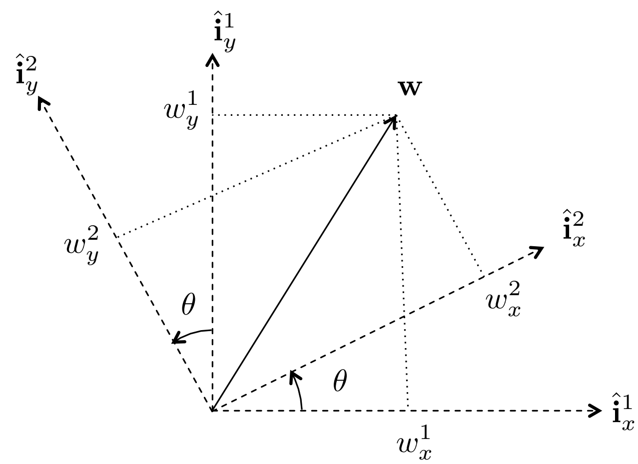

Figure 1: The vector $\mathbf{w}$ can be expressed with respect to two different reference frames.

Figure 1 shows a vector $\mathbf{w}$ and two different coordinate frames with orthogonal axes denoted $\mathcal{F}^1 = \{\mathbf{i}_x^1, \mathbf{i}_y^1, \mathbf{i}_z^1\}$ for Frame 1, and $\mathcal{F}^2 = \{\mathbf{i}_x^2, \mathbf{i}_y^2, \mathbf{i}_z^2\}$ for Frame 2, where $\mathbf{i}_*^*$ are unit vectors in the direction of the axes. The two frames are centered at the same location, but frame $\mathcal{F}^2$ is rotated with respect to frame $\mathcal{F}^1$ by a right handed rotation about the $\mathbf{i}_z^1$-axis of θ, where $\mathbf{i}_z^1 = \mathbf{i}_z^2$. Let $\mathbf{w}$ be a vector that lies entirely in the $x - y$ plane of both reference frames. The vector $\mathbf{w}$ is expressed with respect to frame $\mathcal{F}^i$ by a three dimensional vector

$$\mathbf{w}^i = \begin{pmatrix} w_x^i \\ w_y^i \\ w_z^i \end{pmatrix},$$

where the superscript i denotes that $\mathbf{w}$ is expressed with respect to $\mathbf{F}^i$, and where w_*^i is the projection of $\mathbf{w}$ along the unit vector $\hat{\mathbf{i}}_*^i$, i.e.,

$$w_*^i = \mathbf{w} \cdot \mathbf{i}_*^i.$$

Referring to Figure 1, $\mathbf{w}$ can be expressed with respect to $\mathcal{F}^1$ as

$$\mathbf{w}^1 = \begin{pmatrix} w_x^1 \\ w_y^1 \\ 0 \end{pmatrix},$$

and with respect to $\mathcal{F}^2$ as

$$\mathbf{w}^2 = \begin{pmatrix} w_x^2 \\ w_y^2 \\ 0 \end{pmatrix}.$$

To derive a relationship between $\mathbf{w}^1$ and $\mathbf{w}^2$ note that

$$\mathbf{w} = w_x^1 \mathbf{i}_x^1 + w_y^1 \mathbf{i}_y^1 = w_x^2 \mathbf{i}_x^2 + w_y^2 \mathbf{i}_y^2.$$

Taking the inner product of $\mathbf{w}$ with $\mathbf{i}_x^2$ gives

$$\mathbf{w} \cdot \mathbf{i}_x^2 = w_x^1 \mathbf{i}_x^1 \cdot \mathbf{i}_x^2 + w_y^1 \mathbf{i}_y^1 \cdot \mathbf{i}_x^2 = w_x^2$$

where we have used the fact that $\mathbf{i}_x^2 \cdot \mathbf{i}_x^2 = 1$ and $\mathbf{i}_y^2 \cdot \mathbf{i}_x^2 = 0$. Similarly, taking the inner product of $\mathbf{w}$ with $\mathbf{i}_y^2$ we get

$$\mathbf{w} \cdot \mathbf{i}_y^2 = w_x^1 \mathbf{i}_x^1 \cdot \mathbf{i}_y^2 + w_y^1 \mathbf{i}_y^1 \cdot \mathbf{i}_y^2 = w_y^2.$$

Expressing these relationships in matrix form, we get

$$\begin{pmatrix} w_x^2 \\ w_y^2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{i}_x^1 \cdot \mathbf{i}_x^2 & \mathbf{i}_y^1 \cdot \mathbf{i}_x^2 & 0 \\ \mathbf{i}_x^1 \cdot \mathbf{i}_y^2 & \mathbf{i}_y^1 \cdot \mathbf{i}_y^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_x^1 \\ w_y^1 \\ 0 \end{pmatrix}. \tag{1}$$

From Figure 1 it can be seen that

$$\begin{aligned}\mathbf{i}_x^1 \cdot \mathbf{i}_x^2 &= \cos\theta \\ \mathbf{i}_y^1 \cdot \mathbf{i}_x^2 &= \sin\theta \\ \mathbf{i}_x^1 \cdot \mathbf{i}_y^2 &= -\sin\theta \\ \mathbf{i}_y^1 \cdot \mathbf{i}_y^2 &= \cos\theta.\end{aligned}$$

Therefore Equation (1) can be written as

$$\mathbf{w}^2 = R(\theta)\mathbf{w}^1$$

where the rotation matrix $R(\theta)$ is given by

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2 Kinematics for Three Wheel Mobile Robot

The geometry of a three wheeled robot is shown in Figure 2.

The vectors $\mathbf{r}_i^b = (r_{xi}^b, r_{yi}^b, 0)^\top$ denote the position of the center of the i^{th} wheel expressed with respect to a reference frame fixed in the body of the robot. The unit vectors $\hat{\mathbf{s}}_i^b = (s_{xi}^b, s_{yi}^b, 0)^\top$ are unit vectors that point in the direction of spin of the i^{th} wheel. The vector $\mathbf{v}^b = (v_x^b, v_y^b, 0)^\top$ denotes the linear velocity of the center of the body fixed reference frame, expressed in $\mathcal{F}^b$, and the vector $\boldsymbol{\omega} = (0, 0, \omega)^\top$ denotes the angular velocity vector of the robot. We will assume that each wheel has radius R, and that the angular speed of the i^{th} wheel is given by Ω_i.

The objective of this section is to derive the relationship between the velocity $\mathbf{v}^b$ and ω and the wheel speeds Ω_i. Let $\mathbf{v}_i$ be the linear velocity vector of the center of the i^{th} wheel. Then $\mathbf{v}_i$ is related the velocity of the center of the robot, and the angular velocity by the expression

$$\mathbf{v}_i = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}_i, \tag{2}$$

where $\times$ denotes the cross product. The linear speed of the i^{th} wheel is the projection of $\mathbf{v}_i$ along the rolling direction of the wheel. Therefore

$$R\Omega_i = \mathbf{v}_i \cdot \hat{\mathbf{s}}_i. \tag{3}$$

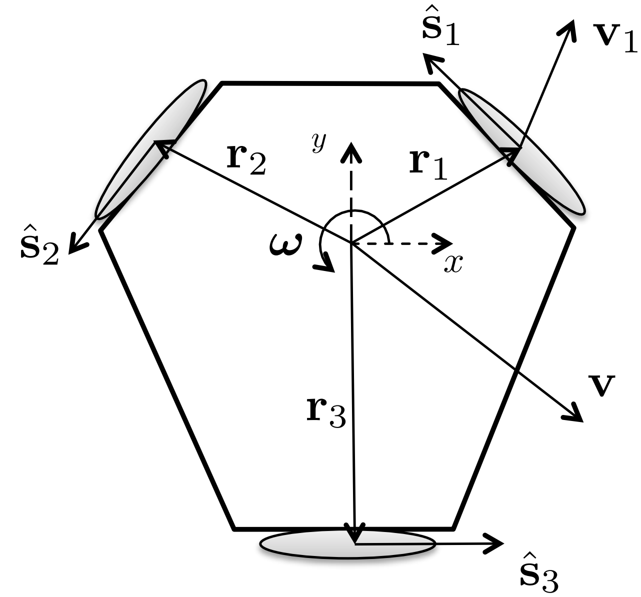

Figure 2: Geometry for a three wheeled mobile robot.

If we express all vectors with respect to the body frame, then we have

$$\begin{aligned}
R\Omega_i &= \mathbf{v}_i^b \cdot \hat{\mathbf{s}}_i^b \\
&= \left[\begin{pmatrix} v_x^b \\ v_y^b \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \begin{pmatrix} r_{xi}^b \\ r_{yi}^b \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} s_{xi}^b \\ s_{yi}^b \\ 0 \end{pmatrix} \\
&= s_{xi}^b v_x^b - s_{xi}^b \omega r_{iy}^b + s_{yi}^b v_y^b + s_{yi}^b \omega r_{xi}^b \\
&= \begin{pmatrix} s_{xi}^b & s_{yi}^b & (s_{yi}^b r_{xi}^b - s_{xi}^b r_{yi}^b) \end{pmatrix} \begin{pmatrix} v_x^b \\ v_y^b \\ \omega \end{pmatrix}.
\end{aligned}$$

Therefore, the forward kinematic relationship between the inertial speeds expressed in the body frame, and the angular speeds of the wheels as

$$\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = M \begin{pmatrix} v_x^b \\ v_y^b \\ \omega \end{pmatrix},$$

where

$$M = \frac{1}{R}\begin{pmatrix} s^b_{x1} & s^b_{y1} & (s^b_{y1}r^b_{x1} - s^b_{x1}r^b_{y1}) \\ s^b_{x2} & s^b_{y2} & (s^b_{y2}r^b_{x2} - s^b_{x2}r^b_{y2}) \\ s^b_{x3} & s^b_{y3} & (s^b_{y3}r^b_{x3} - s^b_{x3}r^b_{y3}) \end{pmatrix}.$$

Similarly, the reverse kinematic relationship is given by

$$\begin{pmatrix} v^b_x \\ v^b_y \\ \omega \end{pmatrix} = M^{-1}\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix}.$$

However, we typically would like to command velocities in the world frame rather than the inertial frame. Note that since $\mathbf{i}^b_z = \mathbf{i}^w_z$, and the angular speed ω is the projection of $\boldsymbol{\omega}$ on $\mathbf{i}^*_z$, that we can express the relationship between world frame velocities and body frames velocities as

$$\begin{pmatrix} v^b_x \\ v^b_y \\ \omega \end{pmatrix} = R(\theta)\begin{pmatrix} v^w_x \\ v^w_y \\ \omega \end{pmatrix},$$

where θ is the heading angle of the robot, or in other words, the angle between $\mathbf{i}^w_x$ and $\mathbf{i}^b_x$.

Therefore, the forward kinematic relationship between speeds in the world frame and wheel speeds is given by

$$\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = MR(\theta)\begin{pmatrix} v^w_x \\ v^w_y \\ \omega \end{pmatrix},$$

and the inverse kinematic relationship is given by

$$\begin{pmatrix} v^w_x \\ v^w_y \\ \omega \end{pmatrix} = R^\top(\theta)M^{-1}\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix},$$

where we have used the fact that $R^{-1}(\theta) = R^\top(\theta)$.

3 Velocity Control for Three Wheel Mobile Robot

The velocity control scheme for a three wheel mobile robot is shown in Figure 3, where the commanded speeds expressed in the world frame are given by v^{wc}_x, v^{wc}_y, and ω^c.

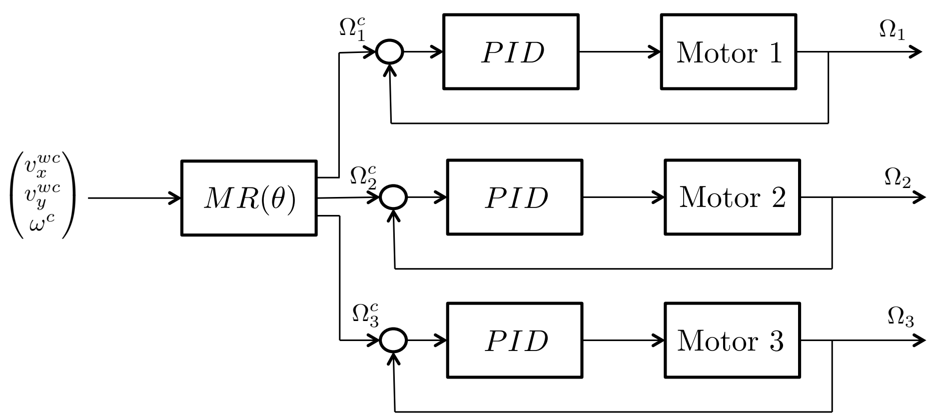

Figure 3: Velocity control scheme for a three wheeled mobile robot.