CHAPTER 1

Solution to Problem 1.1. We have

$$A = \{2, 4, 6\}, \qquad B = \{4, 5, 6\},$$

so $A \cup B = \{2, 4, 5, 6\}$, and

$$(A \cup B)^c = \{1, 3\}.$$

On the other hand,

$$A^c \cap B^c = \{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}.$$

Similarly, we have $A \cap B = \{4, 6\}$, and

$$(A \cap B)^c = \{1, 2, 3, 5\}.$$

On the other hand,

$$A^c \cup B^c = \{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}.$$

Solution to Problem 1.2. (a) By using a Venn diagram it can be seen that for any sets S and T, we have

$$S = (S \cap T) \cup (S \cap T^c).$$

(Alternatively, argue that any x must belong to either T or to T^c, so x belongs to S if and only if it belongs to $S \cap T$ or to $S \cap T^c$.) Apply this equality with $S = A^c$ and $T = B$, to obtain the first relation

$$A^c = (A^c \cap B) \cup (A^c \cap B^c).$$

Interchange the roles of A and B to obtain the second relation.

(b) By De Morgan's law, we have

$$(A \cap B)^c = A^c \cup B^c,$$

and by using the equalities of part (a), we obtain

$$(A\cap B)^c = \big((A^c\cap B)\cup(A^c\cap B^c)\big)\cup\big((A\cap B^c)\cup(A^c\cap B^c)\big) = (A^c\cap B)\cup(A^c\cap B^c)\cup(A\cap B^c).$$

(c) We have $A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$, so $A \cap B = \{1, 3\}$. Therefore,

$$(A \cap B)^c = \{2, 4, 5, 6\},$$

and

$$A^c \cap B = \{2\}, \qquad A^c \cap B^c = \{4, 6\}, \qquad A \cap B^c = \{5\}.$$

Thus, the equality of part (b) is verified.

Solution to Problem 1.5. Let G and C be the events that the chosen student is a genius and a chocolate lover, respectively. We have $\mathbf{P}(G) = 0.6$, $\mathbf{P}(C) = 0.7$, and $\mathbf{P}(G \cap C) = 0.4$. We are interested in $\mathbf{P}(G^c \cap C^c)$, which is obtained with the following calculation:

$$\mathbf{P}(G^c \cap C^c) = 1 - \mathbf{P}(G \cup C) = 1 - \big(\mathbf{P}(G) + \mathbf{P}(C) - \mathbf{P}(G \cap C)\big) = 1 - (0.6 + 0.7 - 0.4) = 0.1.$$

Solution to Problem 1.6. We first determine the probabilities of the six possible outcomes. Let $a = \mathbf{P}(\{1\}) = \mathbf{P}(\{3\}) = \mathbf{P}(\{5\})$ and $b = \mathbf{P}(\{2\}) = \mathbf{P}(\{4\}) = \mathbf{P}(\{6\})$. We are given that $b = 2a$. By the additivity and normalization axioms, $1 = 3a + 3b = 3a + 6a = 9a$. Thus, $a = 1/9$, $b = 2/9$, and $\mathbf{P}(\{1, 2, 3\}) = 4/9$.

Solution to Problem 1.7. The outcome of this experiment can be any finite sequence of the form $(a_1, a_2, \ldots, a_n)$, where n is an arbitrary positive integer, $a_1, a_2, \ldots, a_{n-1}$ belong to $\{1, 3\}$, and a_n belongs to $\{2, 4\}$. In addition, there are possible outcomes in which an even number is never obtained. Such outcomes are infinite sequences $(a_1, a_2, \ldots)$, with each element in the sequence belonging to $\{1, 3\}$. The sample space consists of all possible outcomes of the above two types.

Solution to Problem 1.8. Let p_i be the probability of winning against the opponent played in the ith turn. Then, you will win the tournament if you win against the 2nd player (probability p_2) and also you win against at least one of the two other players [probability $p_1 + (1 - p_1)p_3 = p_1 + p_3 - p_1 p_3$]. Thus, the probability of winning the tournament is

$$p_2(p_1 + p_3 - p_1 p_3).$$

The order $(1, 2, 3)$ is optimal if and only if the above probability is no less than the probabilities corresponding to the two alternative orders, i.e.,

$$p_2(p_1 + p_3 - p_1 p_3) \geq p_1(p_2 + p_3 - p_2 p_3),$$

$$p_2(p_1 + p_3 - p_1 p_3) \geq p_3(p_2 + p_1 - p_2 p_1).$$

It can be seen that the first inequality above is equivalent to $p_2 \geq p_1$, while the second inequality above is equivalent to $p_2 \geq p_3$.

Solution to Problem 1.9. (a) Since $\Omega = \cup_{i=1}^n S_i$, we have

$$A = \bigcup_{i=1}^{n} (A \cap S_i),$$

while the sets $A \cap S_i$ are disjoint. The result follows by using the additivity axiom.

(b) The events $B \cap C^c$, $B^c \cap C$, $B \cap C$, and $B^c \cap C^c$ form a partition of Ω, so by part (a), we have

$$\mathbf{P}(A) = \mathbf{P}(A \cap B \cap C^c) + \mathbf{P}(A \cap B^c \cap C) + \mathbf{P}(A \cap B \cap C) + \mathbf{P}(A \cap B^c \cap C^c). \quad (1)$$