Mathematical Models for Mobile Robots

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1 Introduction

The objective of these notes is to collect in one location, a number of models that are useful for mobile robotics, with particular application to robot soccer.

2 DC Motor

The basic actuation device on almost all mobile robots is the DC motor. There are numerous sources that describe how DC motors work. Particularly good introductory material includes [1, Chap. 7] and [2, p.171–183].

A simplified circuit model for a DC motor is shown in Figure 1, where

e_m is the input voltage,

i is the motor current,

R is the motor resistance,

e_{emf} is the back emf due to the spinning motor shaft,

r_2/r_1 is the gear ratio,

J_m is the motor inertia,

J_L is the inertia of the load connected to the shaft,

τ_m and τ_L are the torque exerted by the motor and the torque exerted on the load respectively,

ω_m and ω_L are the angular velocities at the motor (before gears) and the load (after gears),

θ_m and θ_L are the angles of the motor (before gears) and the load (after gears).

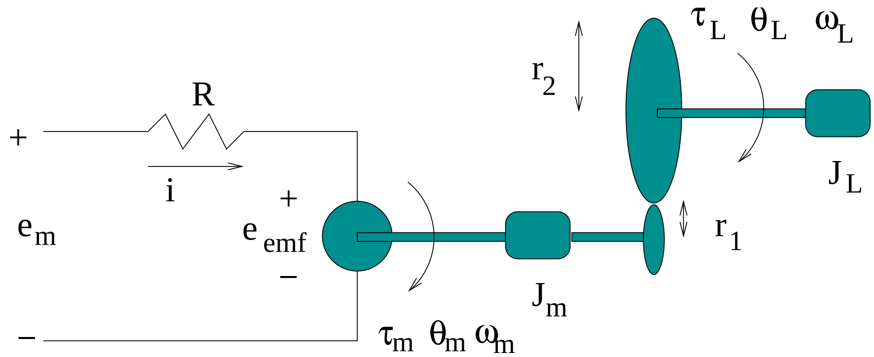

Figure 1: Simplified circuit model for a DC motor.

Let the gear ratio be denoted as

$$k_g \triangleq \frac{r_2}{r_1}.$$

The gear ratio can be determined from the spec sheets for the gear head. The green motors provided in the robot soccer kits have gear ratios of either 19:1 ($k_g = 19$) or 14:1 ($k_g = 14$). The relationship between τ_m and τ_L is given by

$$\tau_L = k_g \tau_m, \tag{1}$$

whereas the relationship between ω_m and ω_L is given by

$$\omega_L = \frac{1}{k_g}\omega_m. \tag{2}$$

The torque exerted by the motor is proportional to the motor current:

$$\tau_m = K_i i, \tag{3}$$

where K_i is the torque constant, which can be determined from the spec sheets of the motor.

The back-emf is proportional to the angular speed of the motor shaft:

$$e_{emf} = K_s \omega_m, \tag{4}$$

where K_s is the speed constant, which can also be determined from the spec sheets of the motor.

Applying Kirkoff's voltage law to the circuit shown in Figure 1 we get

$$e_m = iR + e_{emf}.$$

Using (3) and (4) we get

$$e_m = \frac{R}{K_i}\tau_m + K_s \omega_m.$$

Using (1) and (2) to reflect to the load we get

$$e_m = \frac{R}{k_g K_i}\tau_L + k_g K_s \omega_L. \tag{5}$$

The total inertia seen at the load is given by [3, p.47-50]

$$J = J_L + k_g^2 J_m. \tag{6}$$

By Newton's second law we have

$$J\dot{\omega}_L = \tau_L, \tag{7}$$

which, using (5) gives

$$\begin{aligned}\dot{\omega}_L &= -\frac{k_g^2 K_i K_s}{JR}\omega_L + \frac{k_g K_i}{JR} e_m \\ &= -\frac{k_g^2 K_i K_s}{R(J_L + k_g^2 J_m)}\omega_L + \frac{k_g K_i}{R(J_L + k_g^2 J_m)} e_m.\end{aligned}$$

Define k_1 and k_2 as

$$\begin{aligned}k_1 &= \frac{k_g K_i}{R(J_L + k_g^2 J_m)}, \\ k_2 &= \frac{k_g^2 K_i K_s}{R(J_L + k_g^2 J_m)}.\end{aligned}$$

The motor dynamics, including gear head and load, can therefore be written as

$$\dot{\omega} = -k_2\omega + k_1 e, \tag{8}$$

where e voltage at the motor terminal, and ω is the angular velocity of the output shaft. The constants k_1 and k_2 can be determined by either system identification or by reading values from the spec sheets and computing a reasonable estimate of J_L.

Alternatively, if we consider the current i as the input, then from (7), (3), and (1) we get

$$J\dot{\omega}_L = k_g K_i i,$$

which using (6) gives

$$\dot{\omega}_L = \frac{k_g K_i}{J_L + k_g^2 J_m} i.$$

The challenge is to derive a reasonable estimate of J_L for a mobile robot. A standard wheel assembly, without gears, is shown in Figure 2.

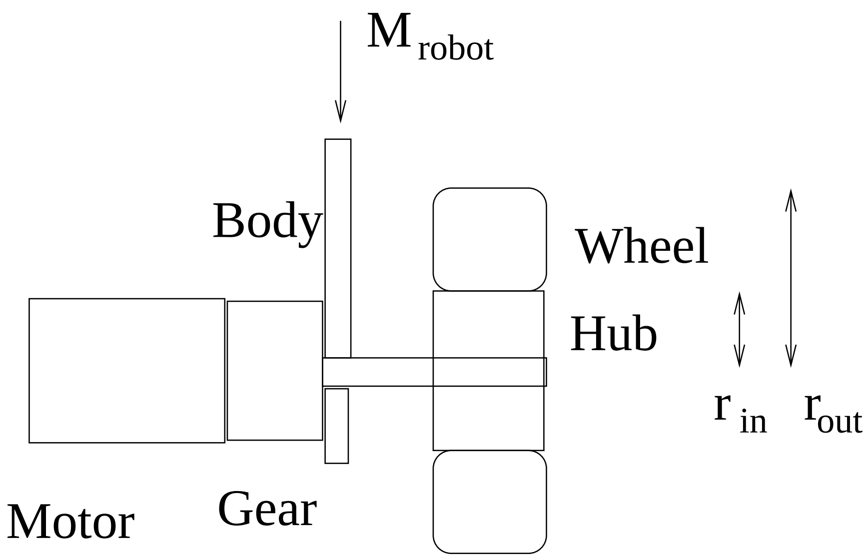

Figure 2: Standard wheel assembly for a mobile robot.

The inertia of the hub and wheel can be calculated from standard formulas [4, p. 238] as

$$J_{hub} = \frac{M_{hub} r_{in}^2}{2}$$
$$J_{wheel} = \frac{M_{wheel}}{2}(r_{in}^2 + r_{out}^2),$$

where M_{hub} and M_{wheel} are mass of the hub and wheel respectively. The mass of the robot exerts a force equal to half the weight of the robot, which is felt at the radius of the wheel. Therefore the inertia due to the mass of the robot is

$$J_{robot} = \frac{M_{robot}}{2} r_{out}^2,$$

where M_{robot} is the mass of the robot. Therefore

$$J_L = J_{hub} + J_{wheel} + J_{robot}.$$

Spec sheets for the motors found in the robot soccer kit are included here: motor specs, gear specs

3 Pulse-Width Modulation

Unfortunately, the MIT handyboard does not produce a voltage e to drive the motor. In fact, most most DC motors are driven with solid state circuits that schematically function as shown in Figure 3[1, p. 218-224]. If switch 1 is on, then full positive current is applied and the motor rotates in the positive direction. On the other hand, if switch 2 is on, then full negative current is applied and the motor rotates in the negative direction. Note that if neither switch 1 or switch 2 is applied then no current flows through the motor. However, as can be seen from Figure 1 there may still be a voltage applied across the terminal of the motor due to the back-emf induced by a spinning shaft. The important point is that the motor is current driven.

The motivation for using a PWM scheme is that they are much easier and cheaper to build then linear voltage amplifiers. The downside is that continuous waveforms cannot be applied to the motor. The input to the motor will look something like Figure 4.

Now suppose that we would like to drive the motor with a voltage signal that looks like Figure **??**. Is it possible to find a bang-off-bang signal that causes the motor to respond in the same way as the signal shown in Figure 5?

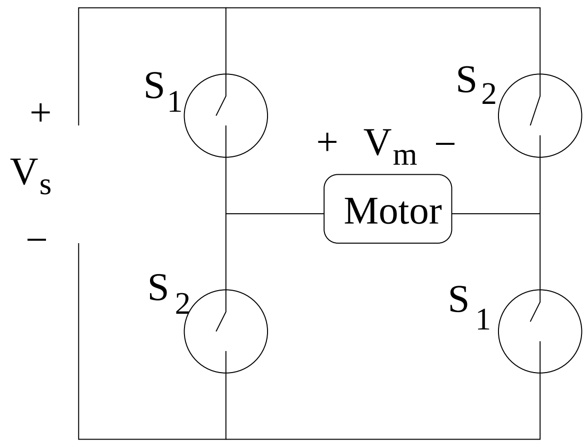

Figure 3: Pulse width modulation circuit.

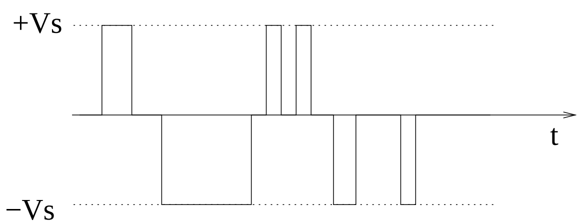

Figure 4: The output of a PWM circuit.

Given a linear system with impulse response $h(t)$, the output is given by

$$y(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau.$$

Therefore the system really responds to the **area** under $u(\tau)$ as opposed to the instantaneous values. If we can find a bang-off-bang signal $m(t)$ with the same area as $u(t)$, i.e.,

$$\int_0^t h(t-\tau)u(\tau)\,d\tau = \int_0^t h(t-\tau)m(\tau)\,d\tau,$$

then we will get identical response.

The motor driver chips on the handyboard have two sample rates: T_1 and $T_2 << T_1$. The user can supply a duty cycle command, $-100 \leq d \leq 100$ at

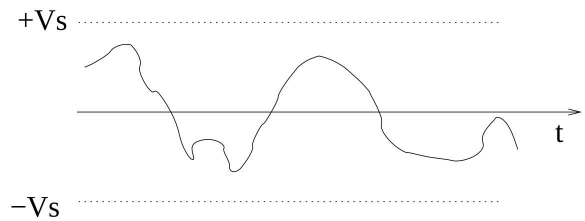

Figure 5: Continuous voltage signal.

the sample rate T_1. The motor driver chip then outputs a discrete voltage waveform such that the area under the waveform is identical to

$$V_s\left(\frac{d}{100}\right),$$

as shown in Figure 6. The sample rate T_1 is the rate at which input commands

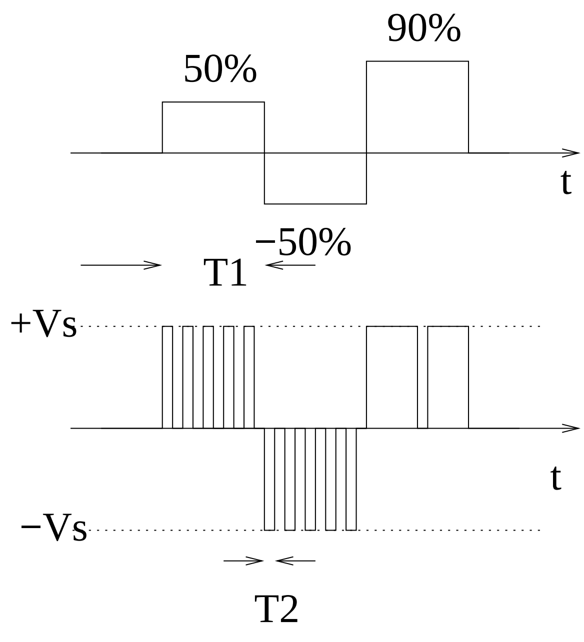

Figure 6: Conversion between PWM and voltage waveforms.

can be issued to the motor driver chip. The sample rate T_2 is the rate at which the voltage at the terminals of the motors is changed. The variable d

is the duty cycle and indicates the percentage of time that the supply voltage is apply to the motor during T_1.

In actuality, the motor driver chips are current sources. When the supply voltage is applied at the terminal of the motor, maximum current is sourced to the motor. When $m(t)$ is zero, no current is supplied to the motor and the voltage at the terminal of the motor will be the back-emf e_{emf}. The voltage at the terminal of the motor will look like Figure 7.

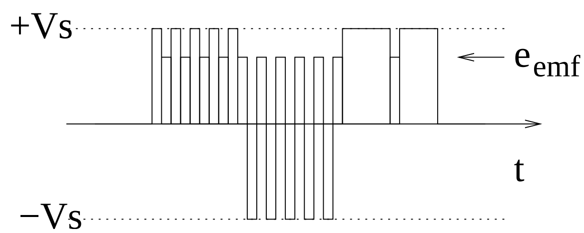

Figure 7: Terminal voltage.

If $d[k]$ is the duty cycle, and $e_{emf}[k]$ is the back-emf then the average voltage during the sample period will be

$$\begin{aligned} e[k] &= V_s\left(\frac{d[k]}{100}\right) + e_{emf}[k]\left(1 - \frac{|d[k]|}{100}\right) \\ &= V_s\left(\frac{d[k]}{100}\right) + K_s k_g \omega[k]\left(1 - \frac{|d[k]|}{100}\right) \\ &\triangleq f(d[k], \omega[k]), \end{aligned}$$

where $\omega[k]$ is the angular speed of the motor shaft.

The output of the motor controller will be $e[k]$, however the motor driver requires $d[k]$. We need to invert $e[k] = f(d[k], \omega[k])$ to find

$$d[k] = f^{-1}(e[k], \omega[k]) \triangleq g(e[k], \omega[k]).$$

Since $d[k] = |d[k]|\ \text{sign}(d[k])$ we can solve for $|d[k]|$ to get

$$|d[k]| = 100\left(\frac{e[k] - K_s k_g \omega[k]}{V_s\ \text{sign}(d[k]) - K_s k_g \omega[k]}\right). \tag{9}$$

We must pick $\text{sign}(d[k])$ such that the right hand side of (9) is nonnegative. We will assume that $e[k] \leq V_s$, which implies that

$$-V_s \leq e[k] \leq V_s,$$

which implies that

$$-V_s - K_s k_g \omega[k] \leq e[k] - K_s k_g \omega[k] \leq V_s - K_s k_g \omega[k].$$

Therefore if $e[k] - K_s k_g \omega[k] \geq 0$, then $V_s - K_s k_g \omega[k] \geq 0$, and we can choose $\text{sign}(d[k]) = +1$. On the other hand, if $e[k] - K_s k_g \omega[k] \leq 0$, then $V_s - K_s k_g \omega[k] \leq 0$, and we can choose $\text{sign}(d[k]) = -1$. Therefore

$$d[k] = g(e[k], \omega[k]) = 100 \left(\frac{e[k] - K_s k_g \omega[k]}{V_s \ \text{sign}(e[k] - K_s k_g \omega[k]) - K_s k_g \omega[k]} \right).$$

4 Batteries

In designing a mobile robot, a key issue is battery life. The purpose of this section is to give a few basic facts about batteries which might help in battery selection. Good reference material for information on batteries is [1, chap 8].

Batteries are rated in Amp-hrs and Volts. Suppose that we have a battery rated at 5 Volts and 1.5 Amp-hrs. Also suppose that we have a 2 kg robot that we would like to operate at a continuous speed of 0.3 m/s. Assume that the coefficient of static friction is $\mu = 0.5$. How long will the batteries last?

The instantaneous power required is

$$P = \mu\, m\, g\, v = 0.5 * (2 \text{ kg}) * (9.8\ m/s^2) * (0.3\ m/s) = 2.94 \text{ Watts}.$$

Recall that

$$\begin{aligned} \text{Watts} &= \frac{\text{Joules}}{\text{sec}} \\ &= \frac{\text{Volts} * \text{Coulombs}}{\text{sec}} \\ &= \frac{\text{Volts} * \text{Amps} * \text{secs}}{\text{sec}} \\ &= \text{Volts} * \text{Amps}, \end{aligned}$$

therefore Volts * Amps * hrs = Watts * hrs. The expected life of the battery is therefore given by

$$T_{battery} = \frac{\text{Volts} * \text{Amp-hrs}}{\text{Power}}.$$

In our case we have

$$T_{battery} = \frac{5 * 1.5}{2.94} = 2.55 \text{ hrs.}$$

Note, however that there are several inaccuracies in the above analysis. First, it is unlikely that the robot will be moving at a constant speed over the entire life of the battery. In addition, the battery will not hold its voltage constant at 5 Volts as it drains. In fact, the voltage level will drop as it drains. However, the above analysis can be used for rough estimates on battery life.

5 Mobile Robot Dynamics

Mobile robot dynamics refer to the relationship between forces, torques and acceleration.

Consider the differential-drive mobile robot shown in Figure 8, where F is the force acting on the robot center of mass, τ is the torque acting about the center of mass, f_r is the force exerted on the robot by the right wheel, f_l is the force exerted on the robot by the left wheel, and b is the distance from the center of mass to the wheels.

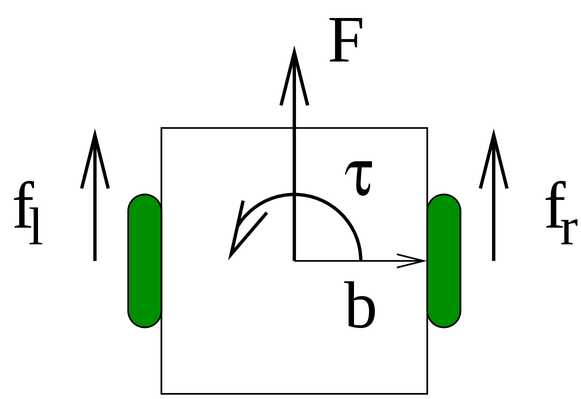

Figure 8: Force model for robot.

If M is the mass of the robot and J is the inertia of the robot, then Newton's second law implies that

$$M\dot{v} = F$$
$$J\dot{\omega} = \tau,$$

where v is the linear speed of the robot and ω is the angular speed. The total force and torque acting on the robot are given by

$$
\begin{aligned}
F &= \frac{f_r + f_l}{2} \\
\tau &= \frac{f_r - f_l}{2b}.
\end{aligned}
$$

Therefore, we can write the relationship between speed and wheel force as

$$
\begin{pmatrix} v \\ \omega \end{pmatrix} = M \begin{pmatrix} f_r \\ f_l \end{pmatrix}
$$

where

$$
M = \begin{pmatrix} \frac{1}{2M} & \frac{1}{2M} \\ \frac{1}{2bJ} & \frac{-1}{2bJ} \end{pmatrix}.
$$

Note that $\det(M) = -\frac{1}{2MbJ} \neq 0$, therefore M is nonsingular and invertible.

The forces f_r can be computed as follows:

$$
\begin{aligned}
f_r &= r\tau_r \\
&= rJ_r\dot{\omega}_r \\
&= rJ_r\left(-k_{2r}\omega_r + k_{1r}e_r\right) \\
&= -rJ_rk_{2r}\omega_r + rJ_rk_{1r}e_r,
\end{aligned}
$$

where r is the radius of the wheel, and J_r is the inertial seen by the shaft of the right wheel. Similarly f_l is given by

$$
f_l = -rJ_lk_{2l}\omega_l + rJ_lk_{1l}e_l.
$$

Combining these expressions into matrix equations we get

$$
\begin{pmatrix} f_r \\ f_l \end{pmatrix} = \mathcal{K}_1 \begin{pmatrix} \omega_r \\ \omega_l \end{pmatrix} + \mathcal{K}_2 \begin{pmatrix} e_r \\ e_l \end{pmatrix},
$$

where

$$
\begin{aligned}
\mathcal{K}_1 &= \begin{pmatrix} -rJ_rk_{2r} & 0 \\ 0 & -rJ_lk_{2l} \end{pmatrix} \\
\mathcal{K}_2 &= \begin{pmatrix} rJ_rk_{1r} & 0 \\ 0 & rJ_lk_{1l} \end{pmatrix}.
\end{aligned}
$$

Since e_r and e_l are known, and ω_r and ω_l can be computed from camera data, f_r and f_l can be computed at every sample period. Alternatively, if force is given by the control system then voltage can be obtained via the equation

$$\begin{pmatrix} e_r \\ e_l \end{pmatrix} = \mathcal{K}_2^{-1} \begin{pmatrix} f_r \\ f_l \end{pmatrix} - \mathcal{K}_2^{-1} \mathcal{K}_1 \begin{pmatrix} \omega_r \\ \omega_l \end{pmatrix}.$$

6 Mobile Robot Kinematics

Kinematics refers to the evolution of the position, and velocity of a mechanical system, without reference to its mass and inertia. Figure 9 shows the inertial coordinate system for a differential-drive mobile robot. The inertial

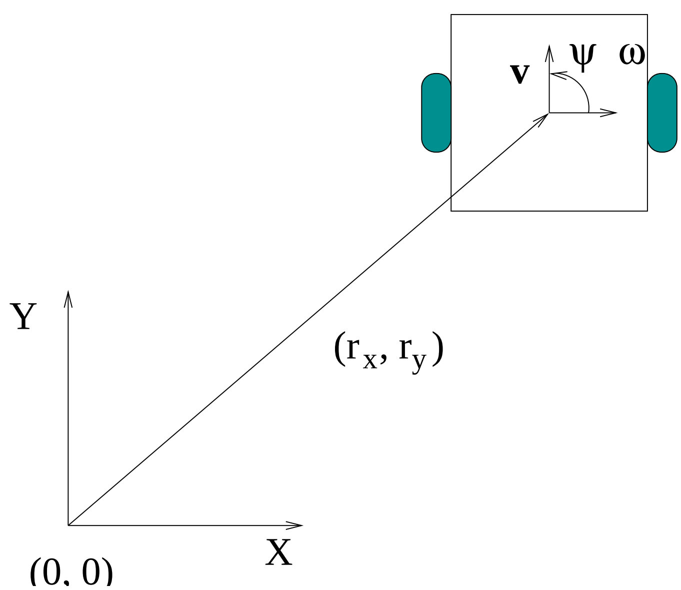

Figure 9: Inertial coordinate frame for a mobile robot.

coordinates of the robot are given by $\mathbf{r} = (r_x, r_y)^T$. The angle that the velocity vector makes with the inertial X axis is given by ψ. The velocity vector of the robot is given by

$$\mathbf{v} = \begin{pmatrix} v\cos(\psi) \\ v\sin(\psi) \end{pmatrix}$$

where v is the linear speed or the robot. Therefore, the differential equations that describe the kinematics of the robot are given by

$$\dot{r}_x = v\cos(\psi) \tag{10}$$

$$\dot{r}_x = v\sin(\psi) \tag{11}$$

$$\dot{\psi} = \omega \tag{12}$$

where the linear speed v and the angular speed ω are assumed to be inputs.

The kinematics given by equations (10)–(12) appear deceptively simple. Note first of all that they are not linear. It is impossible to write equations (10)–(12) as a transfer matrix from $(v(s), \omega(s))^T$ to $(r_x(s), r_y(s), \psi(s))^T$. An attempt to linearize the equations is also futile. Letting $x = (r_x, r_y, \psi)^T$, $u = (v, \omega)^T$, and $f(x, u) = (v\cos(\psi), v\sin(\psi), \omega)^T$, the linearization about (x_0, u_0) is given by

$$\begin{aligned}\dot{\delta x} &= \left.\frac{\partial f}{\partial x}\right|_{(x_0,u_0)} \delta x + \left.\frac{\partial f}{\partial u}\right|_{(x_0,u_0)} \delta u \\ &= \begin{pmatrix} 0 & 0 & -v_0\sin(\psi_0) \\ 0 & 0 & v_0\cos(\psi_0) \\ 0 & 0 & 0 \end{pmatrix} \delta x + \begin{pmatrix} \cos(\psi_0) & 0 \\ \sin(\psi_0) & 0 \\ 0 & 1 \end{pmatrix} \delta u,\end{aligned}$$

which is uncontrollable as $v_0 \to 0$. Actually control strategies can be based on the linearization as long as $v(t) > 0$ for all t.

Note that the kinematics are such that the instantaneous velocity must always point along the vector $(\cos(\psi), \sin(\psi))^T$. In particular, this prevents the robot from moving sideways. Therefore, there is an explicit constraint on the velocity vector of the robot. Constraints of this form are called "nonholonomic." It turns out that many vehicles have nonholonomic kinematics. Examples include mobile robots, cars, airplanes, ships, and submarines. Fortunately for those of us who do research in this area, nonholonomic vehicles are difficult to maneuver automatically. The following two sections will show two mathematical tricks that can be performed on equations (10)–(12) to render simplified models which are linear.

7 Linear Kinematics

The basic idea behind the first trick is extremely simple, but unfortunately not at useful as the second. The idea is to create a master-slave relationship between the two wheels by setting $e_l = \alpha e_r$ where $\alpha \in [-1, 1]$.

Describe how to reduce the order of the kinematics my configuring right and left wheels in a master-slave configuration.

8 Feedback Linearization

It will often be desirable to follow trajectories. If we require that the center of the robot move along the trajectory, then more advanced techniques than those studied in ECEn 483 must be used. However, if we relax the objective to require that the front of the robot follow a trajectory, then we will be able to linearize the dynamics about that point using a technique referred to in the literature as feedback linearization [5].

Consider the schematic of a mobile robot shown in Figure 10. Note that

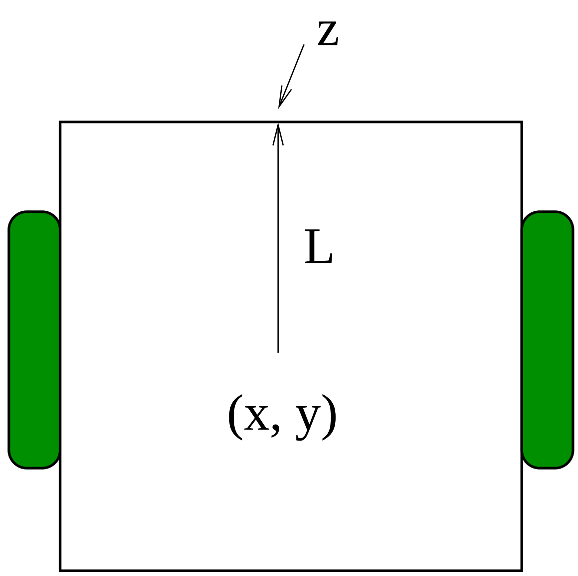

Figure 10: Schematic for feedback linearization.

while the center of the robot cannot move in any direction, and is therefore not small signal controllable, the point at the front of the robot labeled $\mathbf{z}$ in Figure 10 can. The vector $\mathbf{z}$ is referred to as the "hand" position of the robot and is given by

$$\mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix} + L \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \end{pmatrix}.$$

Differentiating $\mathbf{z}$ we get

$$\begin{aligned}
\dot{\mathbf{z}} &= \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + L \begin{pmatrix} -\sin(\psi)\dot{\psi} \\ \cos(\psi)\dot{\psi} \end{pmatrix} \\
&= \begin{pmatrix} v\cos(\psi) - L\sin(\psi)\omega \\ v\sin(\psi) + L\cos(\psi)\omega \end{pmatrix} \\
&= \begin{pmatrix} \cos(\psi) & -L\sin(\psi) \\ \sin(\psi) & L\cos(\psi) \end{pmatrix} \begin{pmatrix} v \\ \omega \end{pmatrix}.
\end{aligned}$$

We can therefore invert the nonlinearity by setting

$$\begin{aligned}
\begin{pmatrix} v \\ \omega \end{pmatrix} &= \begin{pmatrix} \cos(\psi) & -L\sin(\psi) \\ \sin(\psi) & L\cos(\psi) \end{pmatrix}^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\psi) & \sin(\psi) \\ -\frac{1}{L}\sin(\psi) & \frac{1}{L}\cos(\psi) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
\end{aligned}$$

The feedback linearized system therefore becomes

$$\begin{aligned}
z_1 &= u_1 \\
z_2 &= u_2.
\end{aligned}$$

9 Simulated UAV Dynamics

Our research interest in the MAGICC lab include unmanned air vehicles (UAV's). If a UAV is flying a constant altitude, then the kinematics of the UAV can be approximated by equations (10)–(12), with the additional constraints that

$$\begin{aligned}
v_{\min} \leq & v \leq v_{\max} \\
-\omega_{\max} \leq & \omega \leq \omega_{\max}.
\end{aligned}$$

If the UAV flys too slowly, then lift will decrease to the point that the aircraft "drops out of the sky." The thrust capability of the engines on the UAV are dependent on velocity. At a certain velocity, thrust goes to zero. This is very similar to the torque on the motor. As the velocity of the motor increases to the point that the back EMF equals the supply voltage, torque goes to zero. In addition, the heading rate of the UAV, (ω in Eq. (12)) is physically limited by the aerodynamics of the aircraft.

The objective of this section is to show how to simulate UAV dynamics with a differential drive robot. We will make the following assumptions:

A1. The linear speed of the robot is given by

$$v = \frac{r}{2}(\omega_r + \omega_l), \tag{13}$$

and the angular speed of the robot is given by

$$\omega = \frac{r}{2b}(\omega_r - \omega_l), \tag{14}$$

where r is the wheel radius, b is the radius of the wheel axle, ω_r is the angular speed of the right wheel, and ω_l is the angular speed of the left wheel.

A2. The relationship between the angular speed of the wheel and the applied duty cycle is given by

$$\omega = f(d), \tag{15}$$

where ω is the angular speed of the wheel and d is the applied duty cycle. The function f can be determined experimentally by driving the wheel at various duty cycles and observing the angular speed of the wheel after the transients have died out. We should note that this assumption implies that the dynamics of the motors will be ignored.

A3. Duty cycle is limited by

$$\begin{aligned} -100 \leq & d_r \leq 100 \\ -100 \leq & d_l \leq 100, \end{aligned}$$

where d_r is the duty cycle applied to the right wheel and d_l is the duty cycle applied to the left wheel.

Using Eq. (13) and (15) we get

$$d_r = f^{-1}\left(\frac{2v}{r} - f(d_l)\right). \tag{16}$$

Using this expression in Eq. (14) and solving for $f(d_l)$ we get

$$f(d_l) = \frac{v}{r} - \frac{b}{r}\omega,$$

which implies that

$$d_l = f^{-1}\left(\frac{v}{r} - \frac{b}{r}\omega\right). \tag{17}$$

Substituting Eq. (17) into Eq. (16) we obtain

$$d_r = f^{-1}\left(\frac{v}{r} + \frac{b}{r}\omega\right). \tag{18}$$

Using either Eq. (18) or Eq. (17) we can derive the maximum linear speed of the robot. The maximum linear speed is obtained by setting $d_r = d_l = 100$, which will produce $\omega = 0$ and $v = v_{\max}$. Therefore from Eq. (18) we obtain

$$v_{\max} = rf(100). \tag{19}$$

Suppose that instead of moving at velocity $v_{\max}$ we move at velocity $v = \alpha v_{\max}$ where $\alpha \in (0, 1]$. At this new velocity it is possible to determine the maximum heading rate from Eq. (18) by setting $d_r = 100$ and $v = \alpha v_{\max}$. The maximum heading rate is given by

$$\omega_{\max} = \frac{(1-\alpha)}{b} v_{\max}. \tag{20}$$

Therefore, given v, or equivalently, given $\alpha \in (0, 1]$, the maximum heading rate is given by Eq. (20).

UAV dynamics can be simulated by picking α and then treating ω as an input to the system, which then chooses the duty cycles according to Eq. (17) and Eq. (18).

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